EECS 336: Introduction to Algorithms **Lecture 1**
Algorithms for Fibonacci Numbers memoization, repeated-squaring

Reading: Chapter 2 & 3.

Announcements:

- discussion on Piazza
- grading:
  - homework: 25%
  - peer review: 25%
  - midterms: 30%
  - final: 15%
  - participation: 5%
- sections, Mondays, various times.
- homework partners
- Homework plan:
  - assigned thursday, due thursday, work in pairs, graded for accuracy and quality.
  - peer review.
- TAs: Yiding Feng, Isaac Lee, Zhiping Ziu.
- office hours

Algorithms

- algorithms are everywhere. examples:
  - digital computers,
  - parlementary procedure,
  - scientific method,
  - biological processes.
- algorithms design and analysis governs everything.
- good algorithms are closest things to magic.
- course philosophy: no particular algorithm is important.
- course goals: how to design, analize, and think about algorithms.
- we will not cover anything you could figure out on your own.

Algorithms for Fibonacci Numbers

"0, 1, 1, 2, 3, 5, 8, 13, 21, …"

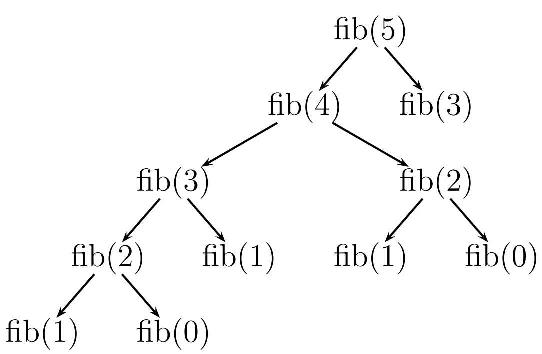
Question: recursive alg?

Algorithm: Recursive Fibonacci

fib(k):

1. if $k \leq 1$ return k
2. (else) return fib$(k-1)$ + fib$(k-2)$

Example:



Analysis

"what is runtime?"

Let $T(k)$ = number of calls to fib

$$
\begin{aligned}
T(0) &= T(1) = 1 \\
T(k) &= T(k-1) + T(k-2) \\
&\geq 2T(k-2) \\
&\geq 2 \times 2T(k-4) \\
&\geq \underbrace{2 \times 2 \times \cdots \times 2}_{k/2 \text{ times}} \times 1 \\
&= 2^{k/2}
\end{aligned}
$$

Conclusion: at least "exponential time"!

Remembering Redundant Computation (memoization)

Idea: remember redundant computation (memoize)

Algorithm: Memoized Recursive Fibonacci

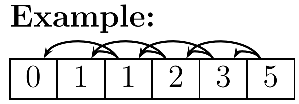
fib-helper(k)

1. if memo[k] ≤ 0
   - memo[k] = fib-helper(k−1) + fib-helper(k−2)
2. return memo[k]

fib(k)

1. memo = new int[k]
2. memo[0] = 0; memo[1] = 1; memo[2,...,k] = -1;
3. return fib-helper(k)

Example:



Analysis

- cost to fill in each entry: 1 additions.
- number of entries: k
- total cost: $T(k) = k$ additions.

Conclusion: "linear time".

Note: memoizing redundant computation is essential part of "dynamic programming".

Iterative Algorithm

Algorithm: Iterative Memoized Fibonacci

fib(k):

1. memo = new int[k];
2. memo[0] = 0, memo[1] = 1
3. for i = 2..k

   memo[i] = memo[i-1] + memo[i-2]
4. return memo[k]

Question: Can we compute fib with less memory (space)?

Algorithm: Iterative Fibonacci

fib(k):

1. last[0] = 0, last[1] = 1;
2. for i = 2..k
   (a) tmp = last[1]
   (b) last[1] = last[0] + last[1]
   (c) last[0] = tmp
3. return last[1]

Question: faster alg?

Fast Fibonacci

Note: algorithm operates on `last` like a matrix multiply

fib(k):

1. $z = [0\ 1]$; $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
2. multiply $z \times \underbrace{A \times A \cdots \times A}_{k-1 \text{ times}}$
3. return $z[1]$

Note: just need to compute $z \times A^{k-1}$

Exponentiation

"compute A^k"

Note: If $k = k_1 + k_2$ then $A^k = A^{k_1} A^{k_2}$

- compute A^{k_1} and A^{k_2} and multiply.
- if $k_1 = k_2$ then redundant computation

Idea: factor $A^k = \left(A^{\texttt{k / 2}}\right)^2 \times A^{\texttt{k \% 2}}$

Algorithm: Repeated Squaring

1. if $k = 1$ return A
2. $k' = \lfloor k/2 \rfloor$.
3. $B =$ repeated-square(A, k').
4. if k odd

   return $B \times B \times A$
5. else

   return $B \times B$

Analysis

Let $T(k)$ = number of multiplies.

$$\begin{aligned} T(1) &= 0 \\ T(k) &= T(k/2) + 2 \\ &= T(k/4) + 2 + 2 \\ &= \underbrace{2 + 2 + 2 \cdots 2}_{\log k \text{ times}} \\ &= 2 \log k \end{aligned}$$

Note: finding subproblems is important part of "divide and conquer"

Algorithm: Fibonacci numbers via repeated squaring

fib(k):

1. $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.
2. $z = [0\ 1] \times$ repeated-square$(A, k-1)$.
3. return $z[1]$.

Analysis

$2 \log k$ 2x2 matrix multiplies.

Conclusions

- runtime analysis
- memoization
- divide and conquer