EECS 336: Introduction to Algorithms **Lecture 2**
Philosophy, Tractibility, Big-Oh

Reading: Chapters 2 & 3.

Announcements:

- Lecture notes on Canvas (before class).
- Practice on "solved problems" in text.
- Prerequisites:
  - EECS 212: Discrete Math.
  - EECS 214: Data Structures.
- Homework:
  - work with partner
  - must communicate solution well.
  - automatically drop 3 lowest hw grades & 3 lowest peer reviews.
  - peer review
    - can you tell if algorithm and proof are correct?
    - communicate algorithms.
  - solutions Thurs, grades Tues.
- Peer review logistics
  - reviews assigned Thurs, due Sun.
  - 3 peer review per problem.
  - 1 peer review is graded (random) (your score close to TA score?)
  - detailed rubric provided.

Last Time:

- motivation
- fibonacci numbers

Today:

- philosophy
- computational tractability
- runtime analysis & big-oh

Algorithms Design and Analysis

gives rigorous mathematical framework for thinking about and solving problems in CS and other fields.

Goals

- quickly compute solutions to problems.
- identify general algorithm design and analysis approaches.
- understand what makes problems tractable or intractable.

Three Steps

1. problem modeling: abstract problem to essential details.
2. algorithm design
3. algorithm analysis
    - efficiency,
    - correctness, and
    - (sometimes) "quality".

Computational Tractability

"is a problem solvable by a computer?"

Def: problem is *tractable* if worst-case runtime to compute solution is polynomial in size of input.

Def: $T(n)$ = worst case runtime of instances of size n.

- size n measured in bits, or
- number of "components".

Example: Fibonacci Numbers

fib(k) has $n = \log k$ bits.

- recursive: $T(n) \approx 2^{2^n}$.
- dynamic program / iterative alg: $T(n) \approx 2^n$.
- repeated squaring: $T(n) \approx n$.

Question: why worst case?

- every instance?
- typical instances?
- random instances?

Question: Benefits?

- usually doable.
- often tight for typical or random instances.
- analyses "compose"

Question: why polynomial?

Answer: polynomial means algorithm scales well, i.e., $T(cn) \leq dT(n)$.

Example:

$$T(n) = n^k$$
$$T(cn) = (cn)^k = \underbrace{c^k}_{d} n^k = dn^k.$$

Tractable vs. Brute-force

- brute-force: "try all solutions, output best one"
- often runtime of brute-force $\geq$ exponential time
- tractable algorithms require exploiting structure of problem.

Main goals for algorithm design

1. show problem is tractable
   exists algorithm with polynomial runtime.
2. show problem is intractable
   for all algorithms, runtime is super-polynomial.

Question: Which is easier?

Answer: showing tractable.

Runtime Analysis

"bound $T(n)$ for algorithm"

Big-Oh Notation

Def: $T(n)$ is $O(f(n))$ if $\exists n_0, c > 0$ such that $\forall n > n_0,\ T(n) < cf(n)$.

Question: why?

Answer:

- exact analysis is too detailed.
- constants vary from machine to machine.

Example:

$$\begin{aligned} T(n) &= an^2 + bn + d \\ &= O(n)?\ O(n^2)?\ O(n^3)? \\ T(n) &\le an^2 + bn^2 + dn^2 \\ &= \underbrace{(a+b+d)}_{c} n^2 \\ &\le cn^3 \end{aligned}$$

Fact 1: $f = O(g) \& g = O(h) \Rightarrow f = O(h)$.

Fact 2: $f = O(h) \& g = O(h) \Rightarrow f + g = O(h)$.

Fact 3: $g = O(f) \Rightarrow g + f = O(f)$.

Proof: (of Fact 2)

$f = O(h) \Rightarrow \exists c, n_0$ such that $\forall n > n_0,\ f(n) < ch(n)$

$g = O(h) \Rightarrow \exists c', n_0'$ such that $\forall n > n_0',\ g(n) < c'h(n)$

$\Rightarrow \forall n > \max(n_0, n_0'),\ f(n) + g(n) \le (c' + c)h(n)$

$\Rightarrow f + g = O(n)$.

QED

Note:

- be succinct: do not write $O(n^2 + n)$, $O(5n)$, etc.
- be tight: if $T(n)$ is n^2 do not say $T(n)$ is $O(n^3)$.

Logarithms and Big-Oh

Def: $\log_b n = \ell \Leftrightarrow b^\ell = n$

- $\log_{10} n$ = number of digits to represent n.
- $\log_2 n$ = number of bits to represent n.

Fact 4: $\forall b, c, \log_b n = O(\log_c n)$

Fact 5: $\forall b, x, \log_b n = O(n^x)$.

Proof: (of Fact 4)

$$\begin{aligned} \log_c n = \ell &\Rightarrow n = c^\ell \\ \log_b n &= \log_b(c^\ell) \\ &= \ell \log_b c \\ &= \log_c n \underbrace{\log_b c}_{d} \\ &= O(\log_c n) \end{aligned}$$

Common Runtimes

$O(\log n)$ – logarithmic

$O(n)$ – linear

$O(n \log n)$

$O(n^2)$ – quadratic

$O(n^3)$ – cubic

$O(n^k)$ – polynomial

$O(2^n)$ – exponential

$O(n!)$

Lower bounds

Def: $T(n)$ is $\Omega(f(n))$ if $\exists n_0, c > 0$ such that $\forall n > n_0, T(n) > cf(n)$.

Exact bounds

Def: $T(n)$ is $\Theta(f(n))$ if $T(n)$ is $O(f(n))$ and $\Omega(f(n))$.

Graphs

"encode pair-wise relationships"

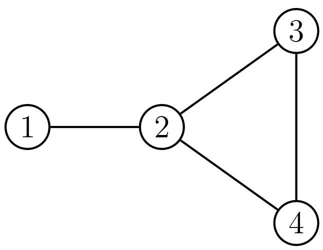
Examples: computer networks, social networks, travel networks, dependencies.

vertices

$G = (V, E)$

edges

Example:



- $V = \{1, 2, 3, 4\}$
- $E = \{(1,2), (2,3), (2,4), (3,4)\}$

Concepts

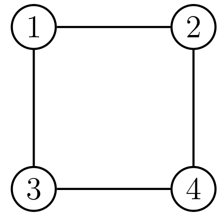
- degree
- neighbors
- paths, path length
- distance
- connectivity, connected components
- directed graphs.

Graph Traversals

"visit all the vertices in a connected component of graph"

- Breadth First Search (BFS).

  Example:

  BFS from 1: 1, 2, 3, 4 or 1, 3, 2, 4.

- Depth First Search (DFS).

  Example: DFS from 1: 1, 2, 4, 3 or 1, 3, 4, 2.