**EECS 336: Introduction to Algorithms** **Lecture 11**

**P vs. NP (cont.)** INDEP-SET, Hamiltonian Cycle

**Reading:** 8.4-8.5; guide to reductions

**Last time:**

- $\mathcal{NP}$-completeness
- 3-SAT $\leq_{\mathcal{P}}$ $INDEP - SET$

**Today:**

- 3-SAT $\leq_{\mathcal{P}}$ INDEP-SET
- 3-SAT $\leq_{\mathcal{P}}$ HC

## Independent Set

**Recall:** INDEP-SET (decision problem)

input: $G = (V, E)$, $k$

output: $S \subset V$

- satisfying $\forall v \in S,\ (u, v) \notin E$
- $|S| \geq k$

**Lemma:** INDEP-SET is $\mathcal{NP}$-hard.

**Proof:** (reduction from 3-SAT)

**Part I:** forward instance construction

convert 3-SAT instance $f$ into INDEP-SET instance $(G, k)$.

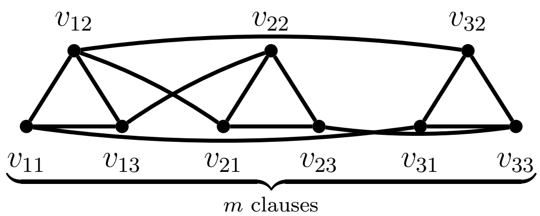
literal $j$ in clause $i$

- vertices $v_{ij}$ correspond to literals $l_{ij}$
- edges for:
  - clause (in triangle)

    "at most one vertex selected per clause"
  - conflicted literals.

    "vertices for conflicting literals cannot be selected"
- "vertex $v_{ij}$ is selected" $\Rightarrow$ "literal $l_{ij}$ is true".
- "indep set of size $m$ $\Leftrightarrow$ "satisfying assignment"

**Example:** $f(z_1, z_2, z_3, z_4) = (z_1 \vee z_2 \vee z_3) \wedge (\bar{z}_2 \vee \bar{z}_3 \vee \bar{z}_4) \wedge (\bar{z}_1 \vee \bar{z}_2 \vee z_4)$




Runtime Analysis: linear time (one vertex per literal).

**Part II::** reverse certificate construction

construct assignment $\mathbf{z}$ from $S$

(if $G$ has indep. set $S$ size $\geq m$ then $f$ is satisfiable.)

(a) For each $z_r$

- if exists nodes in $S$ are labeled by "$z_r$"

  $\Rightarrow$ set $z_r = 1$

- else

  $\Rightarrow$ set $z_r = 0$

**Note:** no two nodes $u, v \in S$ labeled by both $z_r$ or $\bar{z}_r$, if so, there is $(u, v)$ edge so $S$ would not be independent.

(b) $f(\mathbf{z}) = T$:

- $S$ has $|S| = m$

$\Rightarrow$ $S$ has one vertex per clause.

- for caluse $i$:
  - if $v_{ij} \in S$ is not negated, then $i$ is true.
  - if $v_{ij} \in S$ is negated, then $i$ is true.

**Part III::** forward certificate construction

construct independent set $S$ from $\mathbf{z}$

(if $f$ is satisfiable then $G$ has indep. set size $\geq m$.)

- let $S'$ be nodes in $G$ corresponding to true literals.
- if more than one node in $S'$ in same triangle drop all but one.

  $\Rightarrow$ $S$.

- $|S| = m$.
- for all $u, v \in S$,
  - $u$ & $v$ not in same triangle.
  - $l_u$ and $l_v$ both true

    $\Rightarrow$ must not conflict

    $\Rightarrow$ no $(l_u, l_v)$ edge in $G$.

  - so $S$ is independent.

## Problem: Hamiltonian Cycle

input: $G = (V, E)$ (directed)

output: cycle $C$ to visit each vertex exactly once.

**Lemma:** hamiltonian cycle is $\mathcal{NP}$-hard

**Proof:** (reduction from 3-SAT)

**Step 1:** construction

- turn 3-SAT formula $f$ in to graph $G$ with hamiltonian cycle iff $f$ is satisfiable.
- idea: variable = isolated path, right-to-left = true, left-to-right = false.
- idea: clause is node, which needs to be hit by at most one literal being true.
- construction:
  - left-right path per variable.
  - splice in clause nodes.

**Step 2:** runtime.

**Step 3:** correctness.

# TSP

**Lemma 0.1** *TSP is $\mathcal{NP}$-hard.*

**Proof:** *reduction from Hamiltonian Cycle*

- *encode edges with cost 1*
- *encode non-edges with cost $n$.*

$\Rightarrow$ *exists HC iff TSP cost $\leq n$*