EECS 336: Introduction to Algorithms P vs. NP Lecture 14 intractability, NP, decision problems

Reading: 8.0-8.3

Last time:

 \bullet max flow alg / ford-fulkerson

• duality: max flow = min cut

Today:

- tractibility and intractibility
- P and NP
- decision problems
- INDEP-SET, 3-SAT, TSP, NP, CIRCUIT-SAT

Intractibility and NP- output: completeness

"when is a problem intractable?"

Def: \mathcal{P} is the class of problems that can be solved in polynomial time.

$$X \in \mathcal{P}$$
 iff

 \exists polynomial $p(\cdot)$,

 $\exists \text{ alg } \mathcal{A},$

 \forall instances x of X,

 $\Rightarrow \mathcal{A}$ solves x and in time O(p(|x|))

Note: easy to show $X \in \mathcal{P}$, just give \mathcal{A} and prove poly runtime.

Examples: network-flow, matching, interval scheduling, etc.

Three Infamous Problems

Problem 1: Independent Set (INDEP-SET)

input: G = (V, E)

output: $S \subset V$

- satisfying $\forall v \in S, (u, v) \notin E$
- maximizing |S|

Problem 2: Satisfiability (SAT)

input: boolean formula $f(\mathbf{z})$

e.g.,
$$f(\mathbf{z}) = (z_1 \vee \bar{z}_2 \vee x_3) \wedge (z_2 \vee \bar{z}_5 \vee z_6) \wedge \cdots$$

• "Yes" if assignment \mathbf{z} with $f(\mathbf{z}) = T$ exists

e.g.,
$$\mathbf{z} = (T, T, F, T, F, \ldots)$$

• "No" otherwise.

Problem 3: Traveling Salesman (TSP)

input:

- G = (V, E), complete graph.
- $c(\cdot) = \text{costs on edges}$.

output: cycle C that

- passes through all vertices exactly once.
- minimizes total cost $\sum_{e \in C} c(e)$.

No polynomial time algorithm is known for any of these problems!

Theory of Intractability

Goal: formal way to argue that no polynomial time algorithm exists (or "unlikely to exist"), i.e., $X \notin \mathcal{P}$.

Challenge: must show that all algorithms fail!

Idea: to show X is difficult, reduce notoriously hard problem Y to X, i.e., reduce from Y.

Example: to show new problem X is hard, e.g., reduce TSP to X, i.e., reduce from TSP.

Def: \underline{Y} reduces to \underline{X} in polynomial time (notation: $\underline{Y} \leq_{\mathcal{P}} X$ if any instance of Y can be solved in a polynomial number of computational steps and a polynomial number of calls to black-box that solves instances of X.

Consequences of $Y \leq_{\mathcal{P}} X$:

1. if X can be solved in polynomial time then so can Y.

Example: X = network-flow; Y = bipartite matching.

2. if Y cannot be solved in polynomial time then neither can X.

Decision Problems

Goal: show SAT, INDEP-SET, TSP equivalently hard.

Challenge: SAT, INDEP-SET, TSP problem solutions are very different.

Idea: focus on decision version of problem.

Def: A <u>decision problem</u> asks "does a feasible solution exist?"

Example: satisfiability.

Def: an <u>optimization problem</u> asks "what is the min (or max) value of a feasible solution?"

Def: the decision problem X_d for optimization problem X is has input (x, D) = "does instance x of X have a feasible solution with value at most (or at least) D?"

Examples:

INDEP-SET_d: set S with $|S| \ge D$

 SAT_d : **z** such that $f(\mathbf{z}) = T$.

TSP_d: tour C with $\sum_{c \in C} c(e) \leq D$

Deciding is as hard as optimizing

Theorem: $X \leq_{\mathcal{P}} X_d$

Proof: (reduction via binary search)

- given
 - \bullet instance x of X
 - black-box \mathcal{A} to solve X_d
- search(A, B) = find optimal value in [A, B].
 - D = (A + B)/2
 - run $\mathcal{A}(x,D)$
 - if "yes", search(A, D)
 - if "no", $\operatorname{search}(D, B)$

Finding solution is as hard as deciding

Example: satisfiability

- 1. if f is satisfiable $\exists \mathbf{z}$ s.t. $f(\mathbf{z}) = T$
- 2. guess $z_n = T$
- 3. let $f'(z_1, ..., z_{n-1}) = f(z_1, ..., z_{n-1}, T)$
- 4. if f' is satisfiable, repeat (2) on f'
- 5. if f' is unsatisfiable, repeat (2) on $f''(z_1, \ldots, z_{n-1}) = f(z_1, \ldots, z_{n-1}, F)$.

Note: since $X_d =_{\mathcal{P}} X$, we write "X" but we mean " X_d "

A notoriously hard problem

Note: all example problem have short certificates that could easily verify "yes" instance.

Def: \mathcal{NP} is the class of problems that have short (polynomial sized) certificates that can easily (in polynomial time) verify "yes" instances.

Historical Note: $\mathcal{NP} = \underline{\text{non-deterministic}}$ polynomial time

"a nondeterministic algorithm could guess the certificate and then verify it in polynomial time"

Note: Not all problems are in \mathcal{NP} .

E.g., unsatisfiability.

Def:

- Problem \underline{X} is in \mathcal{NP} if exists short easily-verifiable certificate.
- Problem X is \mathcal{NP} -hard if $\forall Y \in \mathcal{NP}$, $Y \leq_{\mathcal{P}} X$.
- Problem X is \mathcal{NP} -complete if $X \in \mathcal{NP}$ and X is \mathcal{NP} -hard.

Lemma: INDEP-SET $\in \mathcal{NP}$.

Lemma: SAT $\in \mathcal{NP}$.

Lemma: $TSP \in \mathcal{NP}$.

Goal: show INDEP-SET, SAT, TSP are

 \mathcal{NP} -complete.

Notorious Problem: NP

input:

- decision problem verifier program VP.
- polynomial $p(\cdot)$.
- \bullet decision problem instance: x

output:

- "Yes" if exists certificate c such that VP(x,c) has "verified = true" at computational step p(|x|).
- "No" otherwise.

Fact: NP is \mathcal{NP} -complete.

Note: Unknown whether $\mathcal{P} = \mathcal{NP}$.

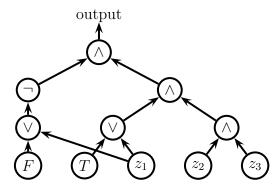
Note: $\leq_{\mathcal{P}}$ is transitive: if $Y \leq_{\mathcal{P}} X$ and

 $X \leq_{\mathcal{P}} Z$ then $Y \leq_{\mathcal{P}} Z$.

Plan: NP $\leq_{\mathcal{P}}$ CIRCUIT-SAT $\leq_{\mathcal{P}}$ SAT.

Circuit Satisfiability

Example:



Problem 4: CIRCUIT-SAT

input: boolean circuit $Q(\mathbf{z})$

- directed acyclic graph G = (V, E)
- internal nodes labeled by logical gates:

• leaves labeled by variables or constants

$$T, F, z_1, \ldots, z_n$$
.

 \bullet root r is output of circuit

output:

- "Yes" if exists **z** with $Q(\mathbf{z}) = T$
- "No" otherwise.

Lemma: CIRCUIT-SAT is \mathcal{NP} -hard.

Proof: (reduce from NP)

- goal: convert NP instance (VP, p, x) to CIRCUIT-SAT instance Q
- $VP(\cdot, \cdot)$ polynomial time

- \Rightarrow computer can run it in poly steps.
- each step of computer is circuit.
- output of one step is input to next step
- unroll p(|x|) steps of computation
 - $\Rightarrow \exists \text{ poly-size circuit } Q'(\mathbf{x}, \mathbf{c}) = VP(x, c)$
- hardcode **x**: $Q(\mathbf{c}) = Q'(\mathbf{x}, \mathbf{c})$
- Conclusion: Q is sat iff exists c with VP(x,c) = "verified".

QED