EECS 336: Introduction to Algorithms	**Lecture 12**
Network Flow	Definition, reduction, bipartite matching

Reading: 7.0-7.5

Last time:

- Shortest-paths (Bellman-Ford Alg)

Today:

- Reductions
- Network flow
- Bipartite matching

Reductions

"to solve problem B given solution to problem A, transform instances from problem B into instances of A, solve, transform solution back"

Problem A: Network Flow

"given a network with bandwidth constraints on links, how much data can we send from source to sink"

Def: a **flow graph** $G = (V, E)$ is a directed graph with:

- $c(e) =$ **capacity** of edge e
- $s \in V$ is **source**.
- $t \in V$ is **sink**.

Def: a **flow** f in G is an assignment of flow to edges "$f(e)$" satisfying:

- capacity: $\forall e,\ f(e) \leq c(e)$
- conservation: $\forall v \neq s, t$,
$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

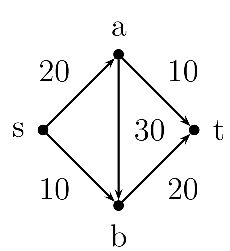
Def: the **value** of a flow is
$$|f| = \sum_{e \text{ out of } s} f(e) = \sum_{e \text{ into } t} f(e)$$

Problem: Network Flow

input: flow graph $G, s, t, c(\cdot)$.

output: flow f with maximum value.

Example:



Max flow = 30.

Theorem 1: there is an algorithm to compute the max flow in polynomial time.

Theorem 2: if capacities are integral, then max flow is integral (on each edge).

Problem B: bipartite matching

Def: $G = (V, E)$ is a bipartite if exists partitioning of V into A and B s.t.,

- $u, v \in A \Rightarrow (u, v) \notin E$,
- $u, v \in B \Rightarrow (u, v) \notin E$,

Recall: a **matching** is a set of edges $M \subseteq E$ each node is connected by at most one edge in M

- a **perfect** matching is one where all nodes are connected by exactly one edge.
- a **maximum** matching is one with maximum cardinality.

Problem: bipartite matching

input: bipartite graph $G = (A, B, E)$

output: a maximum matching M.

Reducing bipartite matching to max flow

"use max flow alg to solve bipartite matching."

Steps:

1. convert matching instance into flow instance.
2. run flow alg flow instance.
3. convert flow soln to matching soln.
4. prove flow soln optimal iff matching soln optimal.

Step 1:

(a) connect s to each $v \in A$ with capacity 1.

(b) connect t to each $u \in B$ with capacity 1.

(c) set capacity of each edge $e \in E$ to 1.

Step 2: compute (integral) max flow f

Step 3: matching is $M = \{e \in E \ : \ f(e) = 1\}$

Step 4: Proof:

- any matching M' can be turned into a flow f' with $|f'| = |M'|$

  (send form s to each matched edge to t one unit of flow)

- any integral flow f' can be turned into a matching M' with $|f'| = |M'|$

  (capacity constraints imply matching)

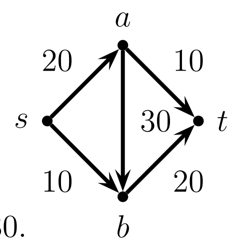
$\Rightarrow$ size of output matching = value of max flow = size of max matching.

Runtime

$T_{\text{matching}}(n, m) = O(n + m) + T_{\text{max flow}}(n, m)$

Network Flow

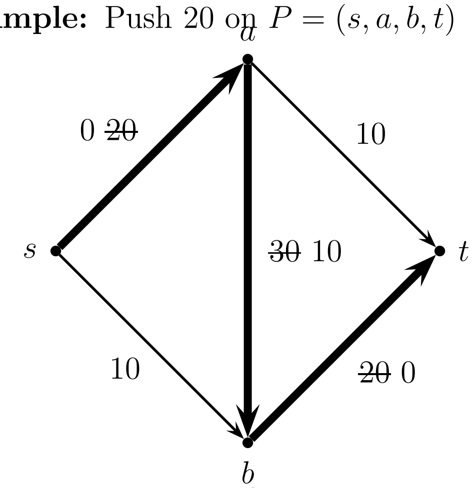
Example:



Max flow = 30.

Idea: repeatedly pus flow on s-t paths until can't push anymore.

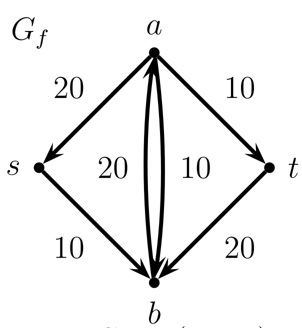
Example: Push 20 on $P = (s, a, b, t)$



Note: when pushing flow, we can undo flow already pushed.

Def: the residual graph G_f for flow f on G is the graph that represents capacity constraints for flows after pushing f

Example: G_f



Construction: $G_f = (V, E_f), c_f(\cdot)$:
For each $e = (u, v) \in E$,

(if $f(e) = c(e)$ discard e)

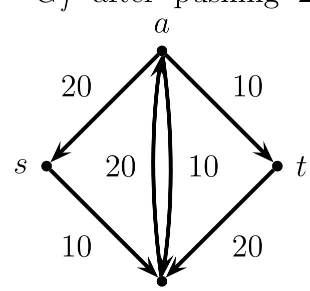
- if $f(e) < c(e)$,
  - add e to E_f
  - $c_f(e) = c(e) - f(e)$
- if $f(e) > 0$
  - let $e' = (v, u)$
  - add e' to E_f
  - $c_f(e') = c(e') + f(e)$

Def: the residual capacity of e in E_f is $c_f(e)$.

Def: the bottleneck capacity of s-t path P in G_f is minimm residual capacity of any edge in P.

Def: an augmenting path P in a residual graph G_f is a path with positive bottleneck capacity.

Example: G_f after pushing 20 on $P = (s, a, b, t)$



Augmenting path $P = (s, b, a, t)$ with bottleneck capacity 10.

Augment f with flow of 10 on P:

- $f(s, b) \leftarrow f(s, b) + 10$
- $f(a, b) \leftarrow f(a, b) - 10$
- $f(a, t) \leftarrow f(a, t) + 10$

Note: can find augmenting paths with BFS.

Algorithm: Augment f with P

- b = bottleneck(P, G_f).

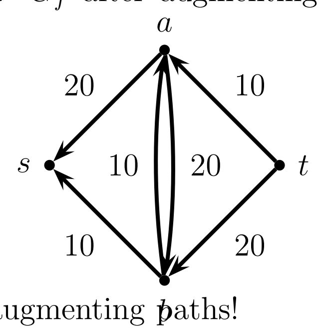
- for e in P:
  - if e a foward edge:

    $f(e) \leftarrow f(e) + b$
  - if e a back edge:

    let $e' =$ back edge

    $f(e') \leftarrow f(e) - b$.

Example: G_f after augmenting with $P = (s, b, a, t)$



No more augmenting paths!

Algorithm: Ford-Fulkerson

- $f \leftarrow$ null flow.
- $G_f \leftarrow G$.
- while exists s-t path P in G (by BFS)
  - augment f with P.
  - $G_f \leftarrow$ residual graph for G and f.
- return f.

Runtime

Each iteration:

- construct G_f: $O(m)$.
- find P: $O(m)$.
- augmentation: $O(n)$.
- (Total: $O(m)$)

Fact: the value of flow increases by bottleneck capacity in each iteration.

Theorem: if C is upper bound on max flow and all capacities are integral then algorithm terminates in $O(C)$ iterations with runtime $O(nC)$

Proof: (by "measure of progress")

1. bottleneck capacities integral:
   - current residual capacities integral

     $\Rightarrow$ integral bottleneck capacity

     $\Rightarrow$ next residual capacities integral
   - induction!
2. bottleneck capacities ≥ 1
3. flow increases by 1 each iteration
4. terminates in $\leq C$ iterations.

QED

Note: $C \leq \sum_{e \text{ out of } s} c(e)$.

Note: Clever choice of augmenting paths gives runtime $O(m^2 \log C)$.

Correctness

1. f is feasible.
2. f is optimal.

Lemma: f is feasible.

Proof: induction!