EECS 336: Introduction to Algorithms	**Lecture 4**
Greedy by Value	MST, Kruskal, matroids

Reading: 4.5-4.6, MIT notes on matroids.

Last Time:

- interval scheduling
- "greedy stays ahead"

Today:

- interval scheduling (cont.)
- greedy-by-value
- minimum spanning trees

Interval Scheduling Recap

"sharing a single resource"

Input:

- n jobs
- one machine
- requests: $\bar{\text{j}}$ob i needs machine between times $s(i)$ and $f(i)$

Goal: schedule to maximize # of jobs scheduled.

Algorithm: Greedy by Min. Finish Time

1. $S = \emptyset$
2. Sort jobs by increasing finish time.
3. For each job j (in sorted order):
   - if j if compatible with S

     schedule j: $S \leftarrow S \cup \{j\}$
   - else discard j

Correctness

"schedule is compatible and optimal"

Lemma 1: schedule of algorithm is compatible

Proof: (by induction, straightforward)

Def:

- let $i_1, \ldots, i_k$ be jobs scheduled by greedy
- let $j_1, \ldots, j_m$ be jobs scheduled by OPT

Goal: show $k = m$.

Approach: "Greedy Stays Ahead"

Lemma 2: for $r \leq k$, $f(i_r) \leq f(j_r)$

Proof: (induction on r)

base case: $r = 0$

- add dummy job 0 with $s(0) = f(0) = -\infty$
- only change: OPT and GREEDY schedule dummy
- so $f(i_0) = f(j_0)$

inductive hypothesis: $f(i_r) \leq f(j_r)$

inductive step:

- Let $I = \{\text{jobs starting after } f(i_r)\}$

  $J = \{\text{jobs starting after } f(j_r)\}$
- IH $\Rightarrow J \subseteq I$
- Alg $\Rightarrow f(i_{r+1}) = \min_{j \in I} f(j)$

  $\leq \min_{j \in J} f(j)$

  $\leq f(j_{r+1})$.

Theorem: Greedy alg. is optimal

Proof: (by contradiction)

- OPT has job j_{k+1} but greedy terminates at k.
- lemma 2 (with $r = k$)

  $\Rightarrow f(i_k) \leq f(j_k)$ (1)
- j_{k+1} is compatible with j_k

  $\Rightarrow f(j_k) \leq s(k_{k+1})$ (2)
- (1)&(2)

  $\Rightarrow f(i_k) \leq s(j_{k+1})$

  $\Rightarrow j_{k+1}$ is compatible with i_k

  $\Rightarrow$ alg doesn't terminate at k

$\rightarrow\leftarrow$

Greedy by Value

"to pick a feasible set with maximum total value"

Example 1: weighted interval scheduling

"if jobs have values"

input:

- n jobs $J = \{1, \ldots, n\}$
- s_i = start time of job i
- f_i = finish time of job i
- v_i = value of job i

output: Schedule $S \subseteq J$ of compatible jobs with maximum total value.

Question: does greedy by finish time work?

Answer: no

```
      1
---------
      ---------
           2
```

Algorithm: Greedy-by-Value

1. $S = \emptyset$
2. Sort elts by decreasing value.
3. For each elt e (in sorted order):

   if $\{e\} \cup S$ is feasible

   add e to S

   else discard e.

Question: does greedy by value work?

Answer: no

```
 1   1   1   1
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 -------------
       2
```

Example 2: minimum spanning tree

"maintaining minimal connectivity in a network, e.g., for broadcast"

input:

- graph $G = (V, E)$
- costs $c(e)$ on edges $e \in E$

output: spanning tree with minimum total cost.

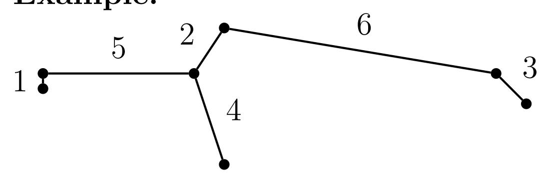
Def: a **spanning tree** of a graph $G = (V, E)$ is $T \subseteq E$ s.t.

(a) (V, T) is connected.

(b) (V, T) is acyclic.

Note: Greedy-by-Value = Kruskal's Alg

Example:



Runtime

$\Theta(m \log n)$

- $\Theta(m \log n)$ to sort.
- check connectivity with union-find data structure

  amortized $O(\log^* n)$ runtime per operation.

  (recall $\ell = \log^* n \Leftrightarrow n = \underbrace{2^{2^{2^{2}}}}_{\ell \text{ times}}$)

  total $O(m \log^* n)$ runtime.

See "MST Structural Observations" at end of notes.

Correctness

"output is tree and has minimum cost"

Lemma 1: Greedy outputs a forest.

Proof: Induction.

Lemma 2: if G is connected, Greedy outputs a tree.

Proof: (by contradiction)

Theorem: Greedy-by-Value is optimal for MSTs

Proof: (by contradiction)

- Greedy and OPT have $n-1$ edges (Fact 1)
- Let $I = \{i_1, \ldots, i_{n-1}\}$ be elt's of Greedy.

  (in order)
- Let $J = \{j_1, \ldots, j_{n-1}\}$ be elt's of OPT.

  (in order)
- Assume for contradiction: $c(I) > c(J)$
- Let r be first index with $c(j_r) < c(i_r)$
- Let $I_{r-1} = \{i_1, \ldots, i_{r-1}\}$
- Let $J_r = \{j_1, \ldots, j_r\}$
- $|I_{r-1}| < |J_r|$ & Augmentation Lemma

  $\Rightarrow$ exists $j \in J_r \setminus I_{r-1}$

  such that $I_{r-1} \cup \{j\}$ is acyclic.
- Suppose j considered after i_k ($k \leq r-1$)
- $I_k \subseteq I_{r-1}$

  $\Rightarrow$ $I_k \cup \{j\} \subseteq I_{r-1} \cup \{j\}$
- $I_{r-1} \cup \{j\}$ acyclic & Fact 2

  $\Rightarrow$ all subsets are acyclic

  $\Rightarrow$ $I_k \cup \{j\}$ acyclic

  $\Rightarrow$ j should have been added.

$\rightarrow\leftarrow$

Structural Observations about MSTs

Def: $G' = (V, E')$ is a **subgraph** of $G = (V, E)$ if $E' \subseteq E$.

Def: An acyclic undirected graph is a **forest**

Def: $A, B \subseteq V$ is a cut if $A \cup B = \emptyset$ and $A \cap B = E$. Edge $e = (u, v)$ crosses cut if $u \in A$ and $v \in B$ (or vice versa).

Fact 1: an MST on n vertices has $n - 1$ edges.

Lemma 1: If $G = (V, F)$ is a forest with m edges then it has $n - m$ connected components.

Proof: Induction (on number of edges)

base case: 0 edges, n CCs.

IH: assume true for m.

IS: show true for $m + 1$

- IH $\Rightarrow n - m$ CCs
- add new edge.
- must not create cycle

$\Rightarrow$ connects two connected components.

$\Rightarrow$ these 2 CCs become 1 CC.

$\Rightarrow$ $n - m - 1$ CCs.

QED

Lemma 2: (Augmentation Lemma) If $I, J \subset E$ are forests and $|I| < |J|$ then exists $e \in J \setminus I$ such that $I \cup \{e\}$ is a forest.

Proof:

Lemma 1

$\Rightarrow$ # CCs of (V, I) > # CCs of $(V, J) \geq$ # CCs of $(V, I \cup J)$

$\Rightarrow$ add elements $e \in J$ to I until # CCs change.

[PICTURE]

$\Rightarrow$ $(V, I \cup \{e\})$ is acyclic.

Fact 2: subgraphs of acyclic graphs are acyclic