CS 332: Online Markets

Lecture 16: Offline Matching

Last Time:

- differential privacy
- exponential mechanism vs exponential weights

Today:

- matching markets
- maximum weight matching
- market clearing
- ascending prices algorithm
- duality

Exercise: House Allocation

Setup:

- two buyers 1 and 2
- two houses A and B
- values:

	Buyer 1	Buyer 2
House A	8	6
House B	7	3

Questions:

- What house does buyer 1 get in the welfare maximizing matching?
- What is the welfare of the optimal matching?

Matching Markets

E.g.

- eBay: sellers and buyers
- ad auctions: advertisers and users
- uber: drivers and riders

Typically:

- one side is long-lived and strategic
- one side is short-lived and behavioral

Setup:

- n buyers (strategic), n items (non-strategic)
- buyers want an item
- items can be sold to a buyer
- buyer i's value for item j: v_j^i
- goal: matching $\mathbf{x}$ to maximize welfare $\sum_{ij} \mathsf{x}_j^i \mathsf{v}_j^i$

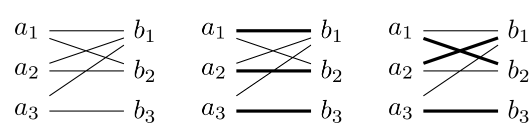
A.k.a.: maximum weighted bipartite maching

Recall: Bipartite Matching

Recall: bipartite graph (A, B, E)

Recall: perfect matching

"$M \subset E$ with each vertex matched exactly once"



Recall:

- **neighborhood** of vertex $a \in A$ is

  $N(a) = \{b \in B : (a, b) \in E\}$

- **neighborhood** of set vertices $S \subset A$ is

  $N(S) = \{b \in B : a \in S \ \& \ (a, b) \in E\}$

Recall Hall's Theorem: a bipartite graph (A, B, E) has perfect matching iff all $S \subset A$ has $|S| \leq |N(S)|$

Market Clearing

"prices where there is no contention for items, and unsold items have price 0"

Def: (bipartite) **demand graph** D at prices $\mathbf{p}$ is:

$N(i) = \text{argmax}_j \, \mathsf{v}^i_j - \mathsf{p}_j$

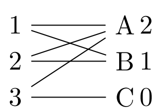
Def: prices $\mathbf{p}$ are **market clearing** if demand graph has perfect matching

Example:

- preferences:

	Buyer 1	Buyer 2	Buyer 3
House A	9	7	6
House B	8	6	2
House C	0	2	4

- prices: $(p_A, p_B, p_C) = (2, 1, 0)$
- demand graph:



Matching Algorithms

Alg: Ascending Prices (AP)

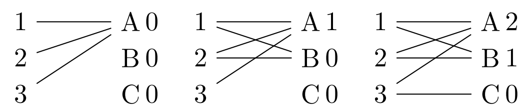
0. initialize prices: $\mathbf{p} = \mathbf{0}$
1. Construct demand graph D
2. if D has perfect matching, output it and halt. (i.e., if $\mathbf{p}$ are market clearing)
3. else,
   a) find set S "minimally" violating Hall's Thm
   b) increase prices of $N(S)$ until demand set of buyer $i \in S$ changes.
   c) repeat (1)

Example:

- input:

	Buyer 1	Buyer 2	Buyer 3
House A	9	7	6
House B	8	6	2
House C	0	2	4

- simulate algorithm:



Exercise: House Pricing

Setup:

- two buyers 1 and 2
- two houses A and B
- values:

	Buyer 1	Buyer 1
House A	8	6
House B	7	3

Questions:

- Are prices $\mathsf{p}_A = 5$ and $\mathsf{p}_B = 3$ market clearing?
- Are prices $\mathsf{p}_A = 7$ and $\mathsf{p}_B = 7$ market clearing?
- What is price for House A in Ascending Prices Algorithm?
- What is price for House B in Ascending Prices Algorithm?

Ascending Prices Analysis

Thm: Ascending Prices Alg maximizes welfare

Proof: Primal = Dual

"for maximization problem, corresponding minimization problem"

Primal Program:

$$\begin{aligned}
\text{Primal}(\mathsf{x}) = \max_{\mathsf{x}} & \sum\nolimits_{ij} \mathsf{v}_j^i\, \mathsf{x}_j^i \\
\text{s.t.} & \sum\nolimits_j \mathsf{x}_j^i \leq 1 \qquad \forall i \qquad (*) \\
& \sum\nolimits_i \mathsf{x}_j^i \leq 1 \qquad \forall j \qquad (**) \\
& \mathsf{x}_j^i \geq 0 \qquad \forall i, j
\end{aligned}$$

Dual Program:

$$\begin{aligned}
\text{Dual}(\mathbf{u}, \mathbf{p}) = \min_{\mathbf{u},\mathbf{p}} & \sum\nolimits_i \mathsf{u}^i + \sum\nolimits_j \mathsf{p}_j \\
\text{s.t.} \ & \mathsf{u}^i + \mathsf{p}_j \geq \mathsf{v}_j^i \qquad \forall i, j \qquad (***) \\
& \mathsf{u}^i \geq 0 \qquad \forall i \\
& \mathsf{p}_j \geq 0 \qquad \forall j
\end{aligned}$$

Intuition:

- utilities $\mathbf{u}$ and prices $\mathbf{p}$
- $\mathsf{u}^i \geq \mathsf{v}_j^i - \mathsf{p}_i$

Lemma 1: $\text{Primal}(\mathbf{x}) \leq \text{Dual}(\mathbf{u}, \mathbf{p})$

Lemma 2: alg's termination condition identifies dual solution with value equal to primal.

Proof 1:

- any primal feasible $\mathbf{x}$
- any dual feasible $\mathbf{u}, \mathbf{p}$

$$\begin{aligned}
\text{Primal}(\mathbf{x}) &= \sum\nolimits_{ij} \mathsf{v}_j^i\, \mathsf{x}_j^i \\
\text{[dual feas. (***)]} \quad &\leq \sum\nolimits_{ij} (\mathsf{u}^i + \mathsf{p}_j)\, \mathsf{x}_j^i \\
&= \sum\nolimits_i \sum\nolimits_j \mathsf{u}^i\, \mathsf{x}_j^i + \sum\nolimits_j \sum\nolimits_i \mathsf{p}_j\, \mathsf{x}_j^i \\
&= \sum\nolimits_i \mathsf{u}^i \sum\nolimits_j \mathsf{x}_j^i + \sum\nolimits_j \mathsf{p}_j \sum\nolimits_i \mathsf{x}_j^i \\
\text{[primal feas. (*,**)]} \quad &\leq \sum\nolimits_i \mathsf{u}^i + \sum\nolimits_j \mathsf{p}_j = \text{Dual}(\mathbf{u}, \mathbf{p})
\end{aligned}$$

Proof 2:

- for prices $\mathbf{p}$ and allocation $\mathbf{x}$ from algorithm,
- set $\mathbf{u}$ as utilities of buyers
- $\mathsf{u}^i = \mathsf{v}_j^i - \mathsf{p}_j$ if $\mathsf{x}_j^i = 1$
- perfect matching of demand sets
  $\Rightarrow \forall i, j:\ \mathsf{u}^i \geq \mathsf{v}_j^i - \mathsf{p}_j$
  $\Rightarrow$ dual feasibility
- inequalities are equalities in proof of Lemma 1.
  $\Rightarrow$ primal = dual.
