CS 396: Online Markets

Lecture 17: Online Matching

Last Time:

- matching markets
- maximum weight matching
- market clearing
- duality

Today:

- duality (cont)
- ascending price mechanism
- externality pricing mechanism
  (a.k.a, Vickrey-Clarke-Groves, VCG)
- online matching

Exercise: Matching Dual

Recall:

$$\begin{aligned} \mathrm{Dual}(\mathbf{u},\mathbf{p}) = \min_{\mathbf{u},\mathbf{p}} \textstyle\sum_i \mathsf{u}_i + \sum_j \mathsf{p}_j & \\ \text{s.t. } \mathsf{u}_i + \mathsf{p}_j \geq \mathsf{v}_{ij} & \quad \forall i,j \\ \mathsf{u}_i \geq 0 & \quad \forall i \\ \mathsf{p}_j \geq 0 & \quad \forall j \end{aligned}$$

Setup:

- two buyers 1 and 2, two houses A and B
- values:

	Buyer 1	Buyer 2
House A	8	6
House B	7	3

Questions: Identify the optimal dual utilities:

- u_1?
- u_2?

Offline Matching Mechainsms

Mech: Externality Pricing (EP)

0. solicit bids $\mathbf{b} = (\mathsf{b}_1^1, \ldots, \mathsf{b}_k^n)$
1. Compute optimal welfare $W = \mathrm{OPT}(\mathbf{b})$ and outcome $\mathbf{x}$
2. Compute optimal welfare without bidder i: $W_{-i} = \mathrm{OPT}(\mathbf{b}_{-i})$
3. Charge bidders **externality**: $\mathsf{p}^i = W_{-i} - (W - \sum_j \mathsf{b}_j^i \mathsf{x}_j^i)$

A.k.a.: Vickrey-Clarke-Groves (VCG) Mechanism

Thm: Externality Pricing Mechanism is truthful.

Proof:

- consider alternative payment $q^i = -(W - \sum_j \mathsf{b}_j^i \mathsf{x}_j^i)$

  "pay bidder value of others"
- truthtelling utility for i is $\sum_j \mathsf{v}_j^i \mathsf{x}_j^i + (W - \sum_j \mathsf{b}_j^i \mathsf{x}_j^i) = W$

  "bidder's utility equals society's welfare"
- EP maximizes society's welfare on truthful bids

  $\Rightarrow$ optimal to bid truthfully.
- q^i is the same as p^i except for "constant" W_{-i}

  "constants don't affect strategies"

Offline Matching Mechanisms (Revisited)

Mech: Ascending Auction (AA)

"implement ascending prices (AP) as auction"

Q: What are good strategies?

A: "report demand sets truthfully"

Thm: EP's prices = AP's prices.

Cor: "truthtelling" is dominant strategy in AA.

Exercise: Externality Pricing

Recall:

- externality pricing mechananism:
  - pick the outcome that maximizes the total welfare.
  - charge each buyer the difference between the optimal welfare without the buyer and the welfare of other buyers (in the optimal welfare outcome)

Setup:

- two buyers 1 and 2, two houses A and B
- bids:

	Buyer 1	Buyer 2
House A	8	6
House B	7	3

Questions:

- Which house does Buyer 2 get in the externality pricing mechanism?
- What is Buyer 2's payment?

Online Matching

"match offline buyers to online items"

Setup:

- n buyers, n items.
- buyer has value v_i for any item in $S_i \subset \{1, \ldots, n\}$
- initially all buyers present
- in round j,
  - item j arrives.
  - match to any remaining buyer i with $S_i \ni j$.
  - matched buyer leaves

Goal: maximize welfare = sum of values of matched buyers.

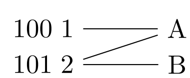
Alg: Greedy Online Matching

- in round j:
  - match to feasible remaining buyer i with highest bid

Q: is this algorithm good?

Example:

- $\mathsf{v}_1 = 100$; $S_1 = \{A\}$
- $\mathsf{v}_2 = 101$; $S_2 = \{A, B\}$



- Greedy:
  - A arrives, assigned to 2.
  - B arrives, not assigned.
- Greedy = 101
- OPT = 201

  $\Rightarrow$ not optimal, but within factor of 2

Q: can it be worse?

A: no.

Thm: Greedy Online Matching is a 2-approximation

Proof: - Approach: (a) each edge of Greedy blocks at most two edges of OPT (b) these blocked edges are lower value

- consider (i, j) matched by OPT
- suppose (i, j) matched by Greedy:
  - charge v_i to (i, j)
- suppose (i, j) not matched by Greedy
- at time j:
  - if i is already matched to j':
    * charge v_i to (i, j')
      (i has the same value for j and j').
  - else, j is already matched to i':
    * when j arrived
    * greedy choose i' instead of i (so $\mathsf{v}_{i'} \geq \mathsf{v}_i$)
    * charge v_i to (i', j)
- each edge in Greedy charged at most twice
- all edges in OPT are accounted for $\Rightarrow$ 2Greedy $\geq$ OPT.

Example:

- $\mathsf{v}_1 = 100$; $S_1 = \{A\}$
- $\mathsf{v}_2 = 101$; $S_2 = \{A, B\}$
- $(1, A)$ not matched by greedy.
  - charged to $(2, A)$ with $\mathsf{v}_2 > \mathsf{v}_1 = 100$
- $(2, B)$ is not matched by greedy.
  - charged to $(2, A)$ with value $\mathsf{v}_2 = \mathsf{v}_2 = 101$
- 2Greedy $= 202 \geq 100 + 101 =$ OPT.

Truthful Online Matching Mechanisms

Note: allocation rule of Greedy Online Matching is monotonic

Mech: Online Greedy Threshold Pricing

- run Online Greedy Algorithm.
- charge buyer minimum bid needed to win.

Thm: Truthtelling is DSE in Online Greedy Threshold Pricing Mechanism

Cor: Online Greedy Threshold is 2-approx in DSE.

Recall: Analysis of Non-truthful Mechanisms

Recall: for distribution of bids, expected critical bid is

$$\hat{\mathsf{B}}_i = \mathbf{E}_{\mathbf{b}}\left[\hat{\mathsf{b}}_i\right]$$

Recall: auction has **conversation ratio** $\mu \geq 1$ if for all distributions of $\mathbf{b}$ and $\mathbf{y} \in \mathcal{X}$:

$$\mu\,\mathbf{E}[\text{Revenue}(\mathbf{b})] \geq \sum\nolimits_i \hat{\mathsf{B}}_i y_i$$

Recall: equilibrium is λ **individually efficient** if

$$\mathsf{u}_i + \hat{\mathsf{B}}_i \geq \lambda\,\mathsf{v}_i$$

Recall: coarse correlated equilibrium is $\lambda = (1 - 1/e) \approx 0.63$ individually efficient

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Non-truthful Online Matching Mechanisms

Mech: Winner-pays-bid Greedy Online Mathching

- run Online Greedy Algorithm on bids.
- charge buyers their bids.

Thm: Winner-pays-bid Greedy Online Matching has conversion ratio $\mu = 1$

Note: winner-pays-bid is better than truthful! (Bidder strategies fix worst-case!)

Proof:

- for feasible $\mathbf{y}$ (agents, items matched at most once):

  $\sum_i \mathsf{b}_i \tilde{x}_i(\mathbf{b}) \geq \sum_i \hat{\mathsf{b}}_i y_i$

- (let M be matching corresponding to $\mathbf{y}$)
- let p_j = "price for item j, or 0 if unsold"
- claim:

  $\sum_i \mathsf{b}_i \tilde{x}_i(\mathbf{b}) \geq^1 \sum_{j:(i,j)\in M} \mathsf{p}_j \geq^2 \sum_{i:(i,j)\in M} \hat{\mathsf{b}}_i$

- $\geq^1$: all revenue $\geq$ some revenue
- $\geq^2$: for $(i, j) \in M$
  - let q = price if item j in market without i
  - $\mathsf{p}_j \geq q$: j sold to highest remaining buyer, adding buyer does not decrease price
  - $q \geq \hat{\mathsf{b}}_i$: i is at least matched to j

    (by outbidding bidder matched to j in market without i)
