EECS 336 - Hardness Reductions - Midterm 2

Practice Midterm

Instructions:

- Write your name on the bottom of every page of the the exam.
- Read each question carefully before answering. If you do not understand a question or notation, ask an instructor before proceeding.
- Wherever possible use the notation defined in the exam instead of inventing your own notation.
- Answer the questions in the spaces provided on the question sheets. Be as succinct as possible. Work on scratch paper first, write final answers in the space provided. Extra scratch paper is available.
- If you make a mistake on one problem clearly cross it out. Do not give multiple answers! If you need to start over on a problem, you can trade in that page for a fresh blank page of the exam.
- If an algorithm is requested, be as succinct as possible. You do not need to supply any proofs unless explicitly asked. All runtimes should be given using big-oh notation.
- The exam has two questions for a total of 100 points.
- You are allowed one handwritten (by you) sheet of notes.
- Detach pages i-iii of the exam, but keep it for your reference during the exam.
- You have 80 minutes.

Note:

- This practice midterm is meant to help students understand the format of questions on the midterm. It is not meant to offer new $\mathcal{NP}$ hardness reductions to practice on. Solutions to these two problems were given in class and in the textbook. The real midterm will not ask you to solve problems you have already seen.
- To best practice for the midterm, practice on the exercises in the back of Chapter 8.

The INDEP-SET decision problem input:

- undirected graph (V, E),
- target independent set size θ.

Output: "yes" if there is a subset S of vertices V with cardinality at least θ and no edge (u, v) in E for any vertices $u, v \in S$.

A *certificate* that easily verifies "yes" instances is the subset S.

Figure 1: INDEP-SET Definition.

Example "yes" and "no" inputs for INDEP-SET are as follows:

	"yes" input	"no" input
(V, E)	$\bullet^{v_1}$— $\bullet^{v_2}$— $\bullet^{v_3}$— $\bullet^{v_4}$	$\bullet^{v_1}$— $\bullet^{v_2}$— $\bullet^{v_3}$— $\bullet^{v_4}$
θ	2	3

Figure 2: INDEP-SET "yes" and "no" inputs.

The VERTEX-COVER decision problem input:

- undirected graph (V, E),
- target cover size θ.

Output: "yes" if there exists a subset S of vertices V that has cardinality at most θ and covers every edge $(u, v) \in E$, i.e., for $(u, v) \in E$ either $u \in S$ or $v \in S$.

A *certificate* that easily verifies "yes" instances is the set S.

Figure 3: VERTEX-COVER Definition.

The 3-SAT decision problem input:

- formula f on n variables with m clauses:
- $f(x_1, \ldots, x_n) = \bigwedge_{j=1}^{m}(l_{j1} \vee l_{j2} \vee l_{j3})$
- where l_{jd} is x_i or $\bar{x}_i$ for $i \in \{1, \ldots, n\}$ and $d \in \{1, 2, 3\}$.

Output: "yes" if there exists an assignment $\mathbf{x} = (x_1, \ldots, x_n)$ with each x_i set to either True or False such that $f(\mathbf{x})$ is True.

A *certificate* that easily verifies "yes" instances is the assignment $\mathbf{x}$ such that $f(\mathbf{x})$ is True.

Figure 4: 3-SAT Definition.

The INDEP-SET decision problem input:

- undirected graph (V, E),
- target independent set size θ.

Output: "yes" if there is a subset S of vertices V with cardinality at least θ and no edge (u, v) in E for any vertices $u, v \in S$.

A *certificate* that easily verifies "yes" instances is the subset S.

Figure 5: INDEP-SET Definition.

Convert 3-SAT instance f to INDEP-SET instance (V^f, E^f, θ^f):

- Vertices $V^f = \{v_{jd} : j \in \{1, \ldots, m\}, d \in \{1, \ldots, 3\}\}$.
- Edges $E^f = \{(v_{jd}, v_{j'd'}) : l_{jd} = \text{“}x_i\text{”} \wedge l_{j'd'} = \text{“}\bar{x}_i\text{”}\}$
- Target independent set size $\theta^f = m$ (the number of clauses).

Figure 6: Incorrect 3-SAT to INDEP-SET forward instance construction

1. Consider proving that VERTEX-COVER is $\mathcal{NP}$-hard by reducing from INDEP-SET. Refer to $\boxed{\text{Figures 3-1 on page ii.}}$

   (a) (25 points) Forward instance construction. Describe how to convert INDEP-SET instance $y = (V, E, \theta)$ to VERTEX-COVER instance $x^y = (V^y, E^y, \theta^y)$ so that y is a yes instance if and only if x^y is a yes instance. (Your construction should be correct and polynomial time, but you do not have to prove it.)

   (b) (5 points) Forward instance construction runtime.

   (c) (10 points) Illustrate your construction on the "yes" and "no" INDEP-SET instances in $\boxed{\text{Figure 2}}$ by giving the VERTEX-COVER instances that your algorithm constructs for each. In both cases, draw and label the resulting graph.

	"yes" instance	"no" instance
(V^y, E^y)		
θ^y		

   (d) (10 points) For the "yes" instances for INDEP-SET in $\boxed{\text{Figure 2}}$ and VERTEX-COVER in your answer to $\boxed{\text{Part (c)}}$, give their certificates.

	INDEP-SET
S	

	VERTEX-COVER
S^y	

Name: ____________________ Question 1: _____/50

2. Consider proving that INDEP-SET is $\mathcal{NP}$-hard by reducing from 3-SAT. Refer to $\boxed{\text{Figures 4-6 on page iii.}}$

   (a) (10 points) Though the *forward instance construction* of $\boxed{\text{Figure 6}}$ is incorrect, it does admit a correct *forward certificate construction*. Give such a *forward certificate construction*, i.e., convert a yes certificate $\mathbf{x}$ of 3-SAT to a yes certificate S^f of INDEP-SET.

   (b) (10 points) Prove that your forward certificate construction is correct.

(Question 2 continues on the next page.)

Name: ____________________ Question 2: _____/20

(c) (10 points) Show that the *forward instance construction* of $\boxed{\text{Figure 6}}$ is incorrect. Specifically, describe a counter example that shows that there does not exist a correct *backward certificate construction* (i.e., a yes certificate S^f of the INDEP-SET instance from the construction can not generally be converted into a yes certificate $\mathbf{x}$ of 3-SAT.

(d) (15 points) Correct the forward instance construction of $\boxed{\text{Figure 6}}$. State the main idea for correcting the construction in English and give a correct construction. (Hint: There is a simple modification that fixes it.)

(e) (5 points) Your construction in $\boxed{\text{Part (d)}}$ is (circle one):

A. Correct. (Part (d) will be graded; credit for this part if (d) is correct.)

B. Incorrect. (Part (d) will not be graded; full credit given for Part (e).)

Name: ____________________ Question 2: _____/30

END OF EXAM