June 4

Friday, June 4, 2021 2:51 PM

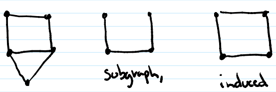
Comp Sci 212

1. Sensitivity Theorem

Announcements

- Last lecture is today
- Office hours until June 9
- Final June 10

Def. Induced subgraph - $G = (V, E)$, $G' = (V', E')$ is an induced subgraph if $V' \subseteq V$, $\{v_1, v_2\} \in E$, and $v_1, v_2 \in V'$, then $\{v_1, v_2\} \in E'$.



subgraph, not induced

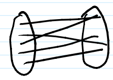
induced subgraph

Hypercube - $G = (\{0,1\}^n, E)$

$E = \{\{v_1, v_2\} \mid v_1 + v_2 \text{ differ in exactly 1 coordinate}\}$.

Sensitivity theorem - If $G = (\{0,1\}^n, E)$, and $G' = (V', E')$ is an induced subgraph s.t. $|V'| > 2^{n-1}$, there is a vertex in G' w/ degree at least $\sqrt{n}$.

Hypercube is bi-partite $\rightarrow$ there exist V_1, V_2 disjoint s.t. $V_1 \cup V_2 = \{0,1\}^n$, all edges go from V_1 to V_2



V_1 V_2

induced graph on vertices from V_1 $\rightarrow$ has no edges

Conjectured by Nisan and Szegedy in 1992

Proved by Hao Huang in 2019

Lot of applications

$\hookrightarrow$ Statement about how powerful quantum computers

Proof. Adjacency matrix - (M_n adjacency matrix $\{0,1\}^n$)

$$M_n = \begin{array}{c c} & \begin{array}{cc} \text{start w/ 0} & \text{start w/ 1} \end{array} \\ \begin{array}{c} \text{start w/ 0} \\ \text{start w/ 1} \end{array} & \begin{bmatrix} M_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & M_{n-1} \end{bmatrix} \end{array}$$

Let $A_0 = 0$

$$A_n = \begin{bmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{bmatrix}$$

signed adjacency matrix (all entries are +1, 0, -1)

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{array}{c c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

$$A_3 \quad \begin{array}{c c} & \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\ \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

Lemma. $A_n^2 = n \cdot I_{2^n}$ (all eigenvalues are $\pm\sqrt{n}$)

Proof. Use Induction.

Base case - $A_0 = 0 \cdot I_1$

Inductive step - Assume $A_n^2 = n \cdot I_{2^n}$

Then A_{n+1}^2

$$A_{n+1} \begin{bmatrix} A_n & I_{2^n} \\ I_{2^n} & -A_n \end{bmatrix}$$

left $\cdot$ top = middle

$$A_{n+1} \begin{bmatrix} A_n & I_{2^n} \\ I_{2^n} & -A_n \end{bmatrix} \begin{bmatrix} A_n^2 + I_{2^n} & A_n - A_n \\ A_n - A_n & I_{2^n} + A_n^2 \end{bmatrix} = A_{n+1}^2$$

$$= \begin{bmatrix} n I_{2^n} + I_{2^n} & 0 \\ 0 & I_{2^n} + n \cdot I_{2^n} \end{bmatrix}$$

$$= (n+1) \begin{bmatrix} I_{2^n} & 0 \\ 0 & I_{2^n} \end{bmatrix} = (n+1) I_{2^{n+1}} \quad \square$$

$$B_n = \begin{bmatrix} A_{n-1} + \sqrt{n}\, I_{2^{n-1}} \\ I_{2^{n-1}} \end{bmatrix} \quad (2^n \times 2^{n-1} \text{ matrix})$$

$$A_n \cdot B_n = \begin{bmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{bmatrix} \begin{bmatrix} A_{n-1} + \sqrt{n}\, I_{2^{n-1}} \\ I_{2^{n-1}} \end{bmatrix} = \begin{bmatrix} A_{n-1}^2 + \sqrt{n}\, A_{n-1} + I_{2^{n-1}} \\ A_{n-1} + \sqrt{n}\, I_{2^{n-1}} - A_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} (n-1) I_{2^{n-1}} + I_{2^{n-1}} + \sqrt{n} \cdot A_{n-1} \\ \sqrt{n}\, I_{2^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{n} \cdot A_{n-1} + n \cdot I_{2^{n-1}} \\ \sqrt{n}\, I_{2^{n-1}} \end{bmatrix}$$

$$= \sqrt{n} \cdot B_n$$

columns of B_n are eigenvectors of A_n

Let x - column vector w/ 2^{n-1} entries

want $(B_n \cdot x)_v = 0$ if $v \notin V'$

2^{n-1} unknowns (each coordinate of x)

$< 2^{n-1}$ equations (handwavy)

more unknowns than equations, there exist non-zero x s.t. $(B_n \cdot x)_v = 0$ if $v \in V'$.

Let $y = B_n \cdot x$. Then,

$$A_n \cdot y = A_n \cdot B_n \cdot x = \sqrt{n} \cdot B_n \cdot x = \sqrt{n} \cdot y$$

Let m be coordinate of y w/ max absolute value.

$$|\sqrt{n} \cdot y_m| = |(A_n y)_m| = \left| \sum_{v \in V} (A_n)_{m,v}\, y_v \right| \qquad y_v = 0 \text{ if } v \notin V'$$

$$= \left| \sum_{v \in V'} (A_n)_{m,v}\, y_v \right| \le \sum_{v \in V'} |(A_n)_{m,v}| \cdot |y_v|$$

$$\le |y_m| \cdot \sum_{v \in V',\ \text{neighbor of } m} 1 = |y_m| \cdot \deg(m) \longrightarrow \sqrt{n} \le \deg(m)$$

↳ in V'

□