June 2

Wednesday, June 2, 2021 2:52 PM

Comp Sci 212

1. Spectral Graph Theory

2. Boolean Hypercube

Announcements

- Final June, 10, practice finals posted
- Office Hours until final

Def. Adjacency matrix A of a graph $G=(V,E)$ is matrix $A \in \mathbb{R}^{V \times V}$,

$$(A)_{i,j} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{o.w.} \end{cases}$$

Thm. If every vertex in $G=(V,E)$ has degree d, $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector w/ eigenvalue d.

Thm. If G is d-regular (every vertex has degree d), and has k connected components, then are k eigenvectors w/ eigenvalue d, and inner product of any two eigenvectors is 0.

Proof. Let $V_1, V_2, \ldots, V_k$ be partition of vertices into connected components.

Let $x^{(i)} \in \mathbb{R}^V$ be $(x^{(i)})_v = \begin{cases} 1 & \text{if } v \in V_i \\ 0 & \text{o.w.} \end{cases}$ ↳ indicator vector

$x^{(i)}$ has degree d

$$(Ax^{(i)})_j = \sum_{v \in V} A_{j,v} \cdot x^{(i)}_v = \sum_{v \in V_i} A_{j,v} = \sum_{\substack{v \in V_i \\ v \text{ is a neighbor of } j}} 1 = \begin{cases} d & \text{if } j \in V_i \\ 0 & \text{o.w.} \end{cases}$$

$A_{j,v}$ = entry in j^{th} row, v^{th} column

$$\begin{array}{c|c} & \begin{matrix} V_1 & V_2 & \ldots & V_k \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ \vdots \end{matrix} & \begin{bmatrix} \square & & 0 \\ & \square & \\ 0 & & \ddots \end{bmatrix} \end{array}$$

$$\langle x^{(i)}, x^{(j)} \rangle = \sum_{v \in V} x^{(i)}_v \cdot x^{(j)}_v = \sum_{v \in V_i, V_j} 1 = 0 \text{ if } V_i \cap V_j = \emptyset$$

Thm. If G is d-regular and bipartite, there is an eigenvector w/ eigenvalue $-d$.

Proof. Let L + R be two parts s.t. edges go from L to R.

Let $x \in \mathbb{R}^V$, $x_v = \begin{cases} 1 & \text{if } v \in L \\ -1 & \text{o.w.} \end{cases}$

$$\begin{array}{c|c} & \begin{matrix} L & R \end{matrix} \\ \begin{matrix} L \\ R \end{matrix} & \begin{bmatrix} 0 & \square \\ \square & 0 \end{bmatrix} \end{array}$$

$v \in L$, $x_v = 1$

$$(Ax)_v = \sum_{w \in V} A_{w,v} \cdot x_w = \sum_{\substack{\text{neighbors } w \\ \text{of } v \\ w \in R}} x_w = \sum_{\substack{\text{neighbors } w \\ \text{of } v}} -1 = -d$$

if $v \in R$, $(Ax)_v = d$

$x_v = -1$

$\square$

Thm. If G is d-regular, all eigenvalues of adjacency matrix are at most d in absolute value.

Proof. Let x be any eigenvector. Let m be the coordinate of x w/ max absolute value, λ is eigenvalue

$$|(Ax)_m| = \left|\sum_{w \in V} A_{w,m} \cdot x_w\right|$$

$$= \left|\sum_{\substack{\text{neighbors } w \\ \text{of } m}} x_w\right| \leq \sum_{\substack{\text{neighbors } w \\ \text{of } m}} |x_w| \leq \sum_{\substack{\text{neighbors } w \\ \text{of } m}} |x_m| = d \cdot |x_m|$$

$$|\lambda \cdot x_m| = |(Ax)_m| \leq d \cdot |x_m|$$

$$|\lambda| \cdot |x_m| \leq d \cdot |x_m| \rightarrow |\lambda| \leq d.$$

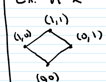
$\square$

Boolean hypercube

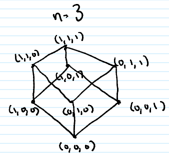
Def. $G = (\{0,1\}^n, E)$

$E = \{ \{v_1, v_2\} \mid v_1 + v_2 \text{ differ in 1 coordinate} \}$.

Ex. $n=2$



$n=3$



Fact. Every vertex has degree n.

Thm. Hypercube is bipartite.

Corollary. Adjacency matrix of hypercube has an eigenvalue of n $-n$, all others are in between.

A_n be adjacency matrix of n-dimensional hypercube

$$A_n = \begin{array}{c|c} & \begin{matrix} V_0 & V_1 \end{matrix} \\ \begin{matrix} V_0 \\ V_1 \end{matrix} & \begin{bmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & A_{n-1} \end{bmatrix} \end{array}$$

V_0 = set of vertices that start w/ 0

V_1 = set of vertices that start w/ 1

$f^{(v)}$, $v \in \{0,1\}^n$, $f^{(v)} \in \mathbb{R}^V$ ($\rightarrow |V| = 2^n$), 1 eigenvector for each $v \in V$

$$f^{(v)}_u = \begin{cases} -1 & \text{if } \langle u,v \rangle \text{ is odd} \\ 1 & \text{if } \langle u,v \rangle \text{ is even} \end{cases} = (-1)^{\langle u,v \rangle}$$

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$$

Ex. If $v = (0, \ldots, 0)$, $\langle u,v \rangle = 0$ for all u, $f^{(v)}$ = all 1's

Thm. $A_n \cdot f^{(v)} = (n - 2 \cdot \# \text{ of 1's in } v) \cdot f^{(v)}$ ← eigenvalue

Proof. $(A_n \cdot f^{(v)})_w = \sum_{\substack{\text{neighbors } u \\ \text{of } w}} f^{(v)}_u$

$w^{\oplus i} = w$, i^{th} coordinate switched (w n-dimensional)

if $w_i = 1 \rightarrow w^{\oplus i}_i = 0$

if $w_i = 0 \rightarrow w^{\oplus i}_i = 1$

$$= \sum_{i=1}^{n} f^{(v)}_{w^{\oplus i}}$$

$$= \sum_{i=1}^{n} (-1)^{\langle v, w^{\oplus i} \rangle}$$

$$= \sum_{\substack{i \text{ s.t.} \\ v_i = 1}} (-1)^{\langle v, w^{\oplus i} \rangle} + \sum_{\substack{i \text{ s.t.} \\ v_i = 0}} (-1)^{\langle v, w^{\oplus i} \rangle}$$

$$= \sum_{\substack{i \text{ s.t.} \\ v_i = 1}} (-1)^{\langle v, w \rangle} \cdot (-1) + \sum_{\substack{i \text{ s.t.} \\ v_i = 0}} (-1)^{\langle v, w \rangle} = \underbrace{(-1)^{\langle v, w \rangle}} \cdot \left(\sum_{\substack{i \text{ s.t.} \\ v_i = 1}} (-1) + \sum_{\substack{i \text{ s.t.} \\ v_i = 0}} 1 \right)$$

$$= f^{(v)}_w \cdot (n - 2 \cdot \# \text{ of 1's in } v)$$

$\square$

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