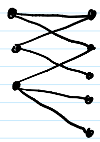
May 24

Monday, May 24, 2021 2:56 PM

Comp Sci 212

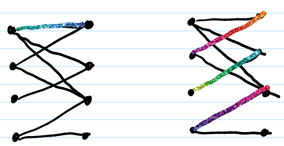
1. Matchings
2. Hall's Theorem
3. Stable Matchings

Def. If G is 2-coloring, it's bipartite

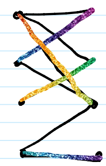


applicants jobs

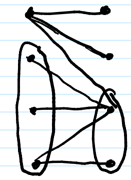
Def. Matching, a subset of edges s.t. every vertex appears in at most one edge.



Def. A perfect matching is a matching s.t. every vertex appears in exactly one edge.



perfect matching



no perfect matching exists

(matchings / perfect matchings also apply to graphs that are not bipartite).

What conditions guarantee a perfect matching exists?

$G = (V, E)$ bipartite $V = L \cup R$

L = set of vertices on left side

R = set on right side

(allow matching to be perfect if all vertices in L are matched)

$N(S) = \{v \mid \{u, v\} \in E, u \in S\}$, set of neighbors of S

The matching condition - If $S \subseteq L$, $|N(S)| \geq |S|$

Theorem. A bipartite graph has a perfect matching iff it satisfies the matching condition

Proof. If a perfect matching exists, let S be any of L. Let $M(S)$ be set of vertices matched with S. $|M(S)| = |S|$, $M(S) \subseteq N(S)$, $|S| = |M(S)| \leq |N(S)|$, and matching condition holds.

Other direction - strong induction on $|L|$

Base case, $|L| = 1$, match $v \in L$ with anything in $N(\{v\})$

Inductive step - Assume statement holds if $|L| \leq k$.

If $|L| = k+1$

Either $|N(S)| = |S|$ for some $S \subseteq L$ ($|S| \leq k$) - Case 2

or $|N(S)| > |S|$ for every S ($|S| \leq k$) - Case 1

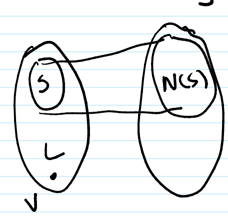
Case 1 - Pick any S subset of L match $v \in S$ with $u \in N(S)$.

Then $G' = (V \setminus \{v, u\}, E$ without edges that contain v or u)

G' satisfies matching condition

(for any $S \subseteq L \setminus \{v\}$, $N(S)$ only decreases by at most 1 from G to G')

Use the matching from the inductive step.

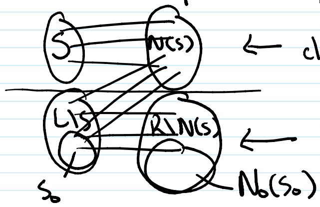


Case 2 - $|S| = |N(S)|$ for some S

match S w/ $N(S)$

$L \setminus S$ w/ $R \setminus N(S)$

both by inductive hypothesis



$\leftarrow$ clearly satisfies matching condition

$\leftarrow$???

Assume you can't match $L \setminus S$ w/ $R \setminus N(S)$.

By inductive hypothesis, there exist $S_0 \subseteq L \setminus S$ s.t.

$\{v \mid \{v,u\} \in E,\ u \in S_0\ \ v \in R \setminus N(S)\} = N_0(S_0)$

set of neighbors of S_0 in $R \setminus N(S)$

$|N_0(S_0)| < |S_0|$

Thus, $N(S \cup S_0) = N_0(S_0) \cup N(S)$

$|N(S \cup S_0)| = |N_0(S_0)| + |N(S)| < |S_0| + |S| = |S_0 \cup S|$ $\longrightarrow$ contradicts matching condition in the original graph

disjoint, sum rule

sum rule