May 21

Friday, May 21, 2021 2:29 PM

Comp Sci 212

1. Colorings
2. Bipartite graphs

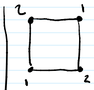
Announcements

- Office hour changes

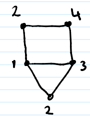
Def. A graph $G=(V,E)$ is k-colorable if there exists a function $f: V \to [k]$ s.t. if $\{u,v\} \in E$, then $f(v) \neq f(u)$

set of k-colors

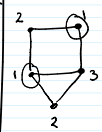
$f \to$ coloring



2-colorable $\chi(G)=2$



4-colorable



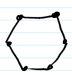
3-colorable $\chi(G)=3$

Def. The chromatic number $\chi(G)$ of a graph G is the smallest k s.t. G is k-colorable.

If G is k-colorable, $\chi(G) \leq k$.

$\chi(G) \leq$ # of vertices in G

Cycle $V=[n]$, $E=\{\{1,2\},\{2,3\},\dots,\{n-1,n\},\{1,n\}\}$



Thm. If $G=([n],E)$ is a cycle on n vertices where n is even, then $\chi(G)=2$.

Proof. G has edges, so $\chi(G)>1$.

G is two-colorable,

Let $f:[n]\to[2]$ be

$$f(x)=\begin{cases}1 & \text{if } x \text{ is odd}\\ 2 & \text{if } x \text{ is even.}\end{cases}$$

If $e \in E$ is of the form $\{i,i+1\}$, one of $i, i+1$ is odd, the other is even, $f(i) \neq f(i+1)$.

If $e=\{1,n\}$, $f(1)=1\neq 2=f(n)$, f is a valid 2-coloring, $\chi(G)\leq 2$. $\square$

Thm. If G is a cycle on n vertices, n is odd, $\chi(G)=3$. (*)

Proof. $\chi(G)>2$ by contradiction.

Assume G is 2-colorable, $f:[n]\to[2]$ be a valid 2-coloring.

Then $f(2)\neq f(1)$, $f(3)\neq f(2)$, therefore $f(1)=f(3)$.

Generally, $f(1)=f(3)=f(5)=\dots=f(n)$, but $\{1,n\}\in E$, contradiction.

G is 3-colorable, $f:[n]\to[3]$

$$f(x)=\begin{cases}1 & \text{if } x \text{ is odd and } x<n\\ 2 & \text{if } x \text{ is even}\\ 3 & \text{if } x=n\end{cases}$$

If e is of the form $\{i,i+1\}$, $i<n$, $f(i)\neq f(i+1)$

$e=\{n-1,n\}$, $f(n-1)=2\neq 3=f(n)$

$e=\{1,n\}$ $f(1)=1\neq 3=f(n)$ $\square$

Complete graph - $V = [n]$, E = all possible edges, $G = (V, E)$ $\chi(G) = n$. All vertices must have distinct colors.

Thm. If $G = (V, E)$ is a graph where all vertices have degree k or less, then G is $k+1$-colorable.
↳ # of neighbors

Proof. Use induction on # of vertices

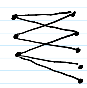
Base case - 1 vertex, no edges, so it's 1-colorable.

Inductive step - Assume theorem is true for $|V|-1$ vertices. Let $v \in V$, $S = \{e \mid e \in E, v \in e\}$ $(\deg(v) = |S|)$
↳ edges that contain v.

Let $G' = (V \setminus \{v\}, E \setminus S)$, has $|V|-1$ vertices, so it's $k+1$-colorable by inductive hypothesis.

Let v have any color different from a neighbor (in G).
This is possible, because v has at most k neighbors.

Def. If G is 2-colorable, then G is bipartite.



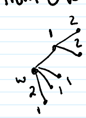
Thm. G is 2-colorable (or bipartite) iff it contains no odd-length cycles.

Proof. If G has an odd-length cycle, it is not 2-colorable.
Theorem (*) → If G has no oddlength cycles

Def. length of a path is the number of edges in a path.

Def. $\text{dist}(u, v)$ = length of shortest path from u to v.

Pick an arbitrary $w \in V$

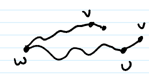


$$f(v) = \begin{cases} 1 & \text{if } \text{dist}(w, v) \text{ is odd} \\ 2 & \text{if } \text{dist}(w, v) \text{ is even} \end{cases}$$

Claim f is a valid 2-coloring

→ Use contradiction, assume $f(u) = f(v)$ for some $\{u, v\} \in E$

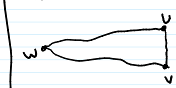
Either, $\text{dist}(w, u) =$ $\text{dist}(w, v)$
$\text{dist}(w, v) + 1$
$\text{dist}(w, v) - 1$



Because $f(u) = f(v)$, $\text{dist}(w, u) = \text{dist}(v, u) = \ell$

$(w, u_1, u_2, u_3, \ldots, u_\ell, u)$
$(w, v_1, v_2, \ldots, v_\ell, v)$ Shortest paths from w to v

Let i be the largest s.t. $u_i = v_i$



Then $(v, v_\ell, v_{\ell-1}, \ldots, \underset{\substack{\| \\ u_i}}{v_i}, u_{i+1}, u_{i+2}, \ldots, u_\ell, u)$

is an odd-length cycle. $\square$