May 17

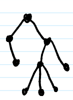
Monday, May 17, 2021 2:56 PM

Comp Sci 212

1. Trees

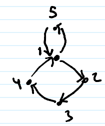
Theorem. The following are equivalent (and define trees)

1. $G=(V,E)$ is connected and acyclic (no subgraph)
2. Every pair of vertices is connected by a unique path (no repeats)
3. G is connected and $|E|=|V|-1$.
4. G is acyclic and $|E|=|V|-1$
5. G is acyclic and adding any new edge creates a cycle.



$|E|=8$
$|V|=9$

Proof. Show that $1\to2$, $2\to3$, $3\to4$, $4\to1$, $5\to1$, $1\to5$



$4\to1$ (Friday)

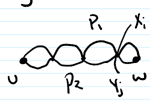
$1\to2$ Use contradiction

Let u, w be two vertices $\to$ not connected, then G is not connected

Assume there's two paths connecting u + w

$(u, x_1, x_2, \ldots, x_n, w)$ - p_1

$(u, y_1, y_2, \ldots, y_m, w)$ - p_2



Let i be the largest number s.t. x_i is in p_2, as y_j. (if no such x_i, use u)

Then $(w, x_n, x_{n-1}, \ldots, \underset{y_j}{x_i}, y_{j+1}, y_{j+2}, \ldots, y_m, w)$ is a cycle.

$2\to3$ Strong induction on $|V|$

Base case - $|V|=1$, obvious

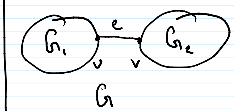
Inductive step - Assume true if $|V|\le k$.

If $|V|=k+1$, let $\{u,v\}\in E$ let $G'=(V, E\setminus\{e\})$

In G', u+v must be disconnected, otherwise there would 2 paths from u to v in G.

Then G' has two connected components, G_1 + G_2

$G_i=(V_i, E_i)$



G_1 + G_2 satisfy 2, so $|E_i|=|V_i|-1$ for $i=1,2$.

$|E|=|E_1|+|E_2|+1=|V_1|-1+|V_2|-1+1=|V|-1$

$|V|=|V_1|+|V_2|$

$3\to4$ Use contradiction

(sketch) Assume G contains a cycle, k edges, k vertices.

Add vertices one by one. Adding each vertex must involve adding an edge, because graph is connected. Ends with $|E|\ge|V|$, contradicts the assumption that $|E|=|V|-1$.

Each time you add a vertex, choose a vertex that neighbors a vertex already in the graph. Such a vertex must exist, otherwise the graph is not connected. Let G' be graph that's built so far, let v be a vertex in G but not G', w be a vertex in G' $(v, x_1, x_2, \ldots, x_n, w)$. Let i be the largest s.t.

x_i is not in G'. Then adding x_i to G' must add an edge, $\{x_i, x_{i+1}\}$ ($\{x_i, w\}$ if $i=n$).

$5\to1$ Use contradiction. Assume G is not connected. In particular, u+v be two vertices not connected. Adding $\{u,v\}$ to E can not create a cycle. Contradicts 5.

$1\to5$ If we add $\{u,v\}$ there is a path from u to v, $(u, v_1, v_2, \ldots, v_n, v)$, adding $\{u,v\}$ turns path into a cycle.