May 14

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Comp Sci 212
1. Degree
2. Connected Components

Def. Graph, $G=(V,E)$, V=vertex set
$E \subseteq \{e \mid e \subseteq V, |e|=2\}$

Def. Degree of a vertex v, $\deg(v)$, $d(v)$ d_v
is $|\{e \mid e \in E, v \in e\}|$, # of edges that contain v

Def. u is a neighbor of v if $\{u,v\} \in E$

Thm. (Handshake Lemma)
$$\sum_{v \in V} \deg(v) = 2|E|$$

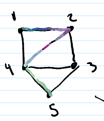
Proof. Each edge contributes 1 to the degree of 2 vertices. □

Proof. $S=\{e \mid e \subseteq V, |e|=2\}$ set of all possible edges.
$$f: S \to \mathbb{R}, \quad f(\{u,v\}) = \begin{cases} 1 & \text{if } \{u,v\} \in E \\ 0 & \text{o.w.} \end{cases}$$
$$\sum_{v \in V} \deg(v) = \sum_{v \in V} \sum_{\substack{u \in V \\ u \neq v}} f(\{u,v\}) = \sum_{\substack{(v,u) \in V \times V \\ v \neq u}} f(\{u,v\})$$
$= 2|E|$, each edge gets counted twice □

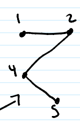
Corollary. Number of vertices of odd degree is even.

Proof. Use contradiction, assume that the number of vertices of odd degree is odd. Then the sum of degrees is odd (odd # of odd + any # of even = odd), contradicts the fact that the sum of degrees $= 2|E|$ □

Def. A path is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ s.t. $\{v_i, v_{i+1}\} \in E$ for all i. (repeats ok)

Ex. 

1-2-4-5
1-4-3
1+5 are connected



Def. $u,v \in V$ are connected if there is a path from u to v.

Def. A graph is connected if all pairs of vertices are connected.

Not connected

Def. Subgraph of $G=(V,E)$ is $G'=(V',E')$
$V' \subseteq V$, $E' \subseteq E$, if $\{u,v\} \in E'$, then $u,v \in V'$.

Ex. a path

Def. Connected component - subgraph containing a vertex v, all vertices connected to v, all edges between such vertices.

connected components

Thm. $G=(V,E)$ has at least $|V|-|E|$ connected components.

Proof. Use induction on number of edges.

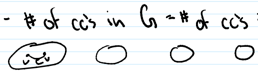
Base case - $|E|=0$, each vertex is in its own connected component, # of cc's is $|V|$.

Inductive step - assume thm is true if $|E|=k$.

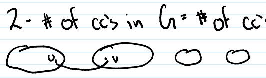
If $|E|=k+1$, consider subgraph $G'=(V, E\setminus\{e\})$ for some e in E. G' has at least $|V|-k$ cc's. (by inductive hypothesis)

$e=\{u,v\}$, $u+v$ are in same cc in G' - Case 1 or different - Case 2

Case 1 - # of cc's in G = # of cc's in $G' = |V|-k \geq |V|-k-1$.



Case 2 - # of cc's in G = # of cc's in $G'-1 = |V|-k-1$

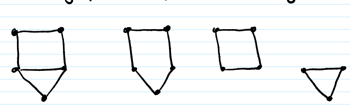


□

Corollary. A connected graph has $|E| \geq |V|-1$.

Proof. If graph is connected, it has 1 connected component, so $|V|-|E| \leq 1$ □

Def. A graph is acyclic if no subgraph is a cycle



G subgraphs of G that are cycles.

Corollary. If a graph is acyclic, it has exactly $|V|-|E|$ cc's.

Proof. It is enough to prove that Case 1 never happens in previous thm.

There must be a path from u to v in G'. $(u, v_1, v_2, \ldots, v_n, v)$. Adding $\{u,v\}$ creates a cycle, contradicting graph is acyclic.

Corollary. If a graph is acyclic and $|E|=|V|-1$, then it is connected (and acyclic).