May 12

Wednesday, May 12, 2021 2:59 PM

Comp Sci 212
1. Chernoff Bounds
2. Graph Theory

Chernoff bounds - (law of large numbers - CS version)

Thm. If $R_1, R_2, \dots, R_n$ are independent random variables, st. $R_i : S \to [-1, 1]$, $\mathbb{E}[R_i] = 0$

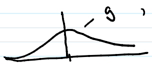
for any $u > 0$

$$\Pr[R_1 + R_2 + \dots + R_n \ge u\sqrt{n}] \le e^{-\frac{u^2}{4e}}$$

Chebyshev - $\le \dfrac{var[R_1 + \dots + R_n]}{u^2 n} = \dfrac{1}{u^2}$

$$var[R_1 + \dots + R_n] = \sum_{i=1}^{n} var[R_i] \le \sum_{i=1}^{n} 1 = n$$

$$\frac{1}{u^2} = \omega\left(e^{-\frac{u^2}{4e}}\right)$$

Gaussian  g, $\mathbb{E}[g] = 0$, $var[g] = n$

$pdf(x) = C_1 e^{-x^2 c_2}$

$c_1, c_2 \to$ constants

number to choose later, $0 < t < 1$.

Prof. $Y = e^{t(R_1 + R_2 + \dots + R_n)}$

$t = \frac{u}{2e\sqrt{n}} < 1$, $u < 2e\sqrt{n}$

$$\Pr[R_1 + \dots + R_n \ge u\sqrt{n}] = \Pr[Y \ge e^{tu\sqrt{n}}] \le \frac{\mathbb{E}[Y]}{e^{tu\sqrt{n}}}$$

$\leftarrow$ need to bound

$$\le e^{t^2 en - tu\sqrt{n}} = e^{\frac{u^2 en}{4e^2 n} - \frac{u^2 \sqrt{n}}{2e\sqrt{n}}} = e^{\frac{u^2}{4e} - \frac{u^2}{2e}} = e^{-\frac{u^2}{4e}} \quad \square$$

$$\mathbb{E}[e^{t(R_1 + \dots + R_n)}] = \mathbb{E}[e^{tR_1}]\,\mathbb{E}[e^{tR_2}] \cdots \mathbb{E}[e^{tR_n}] \le (e^{t^2 e})^n = e^{t^2 en}$$

if $R_1, \dots, R_n$ are independent, then so are $e^{tR_1}, \dots, e^{tR_n}$

$$\mathbb{E}[e^{tR_j}] = \mathbb{E}\left[1 + tR_j + \frac{t^2 R_j^2}{2!} + \frac{t^3 R_j^3}{3!} + \dots\right]$$

(Taylor series)

$$= \mathbb{E}[1] + \overset{0}{\mathbb{E}[tR_j]} + \mathbb{E}\left[\frac{t^2 R_j^2}{2!}\right] + \dots$$

$$= \mathbb{E}[1] + \sum_{i=2}^{\infty} \frac{\mathbb{E}[t^i R_j^i]}{i!} \quad (|R_j| \le 1) \quad (t < 1)$$

$$= 1 + \sum_{i=2}^{\infty} \frac{t^i}{i!} \le 1 + \sum_{i=2}^{\infty} \frac{t^2}{i!}$$

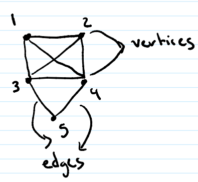
$$= 1 + t^2 \sum_{i=2}^{\infty} \frac{1}{i!} \le 1 + t^2 e \le e^{t^2 e} \quad (1 + x \le e^x)$$

$$\left(e = \sum_{i=0}^{\infty} \frac{1}{i!} \ge \sum_{i=2}^{\infty} \frac{1}{i!}\right)$$

Graphs

Def. (Graph) Vertex set V, Edge set $E \subseteq V \times V$

$G = (V, E)$



$V = \{1, 2, 3, 4, 5\}$

$E = \{(1,2), (1,3), (1,4), (2,1) \dots\}$

Graph can represent

- Social networks V = people E = friendships
- Transportation networks V = stops E = routes
- Job applications V = people, jobs (union) E = applications

Directed graphs $(u,v) \in E$ does not imply $(v,u) \in E$

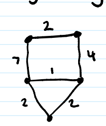
(Instagram) (arrows)

Undirected graphs $(u,v) \in E$ does imply $(v,u) \in E$

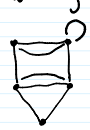
(Facebook) (no arrows)

Weighted graphs $G = (V, E, w)$ $w: E \to \mathbb{R}$

(length of route, strength of friendship)



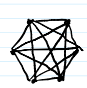
Multi-graph - E is a multi-set



Def. self-loop is an edge of the form (v,v)

Def. simple graph - an undirected graph with no self-loops, not weighted, not a multi-graph

E can consist of subsets of size 2. (rather than pairs)



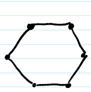
complete graph

E = set of subsets of size 2

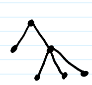
Common graphs, $V = [n]$

path

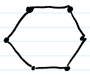
$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \dots \{n-1, n\}\}$



$E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{1, n\}\}$

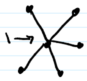


Trees (talk about on Monday)



cycle

$$E = \{\{1,2\}, \{2,3\}, \dots, \{n-1,n\}, \{1,n\}\}$$



star

$$E = \{\{1,2\}, \{1,3\}, \dots, \{1,n\}\}$$