May 10
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Comp Sci 212
1. Chebyshev's Inequality
2. Variance
3. Moment / Chernoff bounds

Markov's Inequality
If R is a non-negative r.v., $x > 0$, then
$$\Pr[R \geq x] \leq \frac{\mathbb{E}[R]}{x} \quad (\text{If } x = \mathbb{E}[R], \leq 1)$$

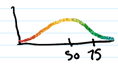
Coins - $S = \{H, T\}^{100}$, $R : S \to \mathbb{R}$, $R(s) = $ # of heads

Sticky - $\Pr[\text{all heads}] = \Pr[\text{all tails}] = \frac{1}{2}$ Uniform

$\mathbb{E}[R] = 50$, for both
$$\Pr[R \geq 75] \leq \frac{50}{75} = \frac{2}{3}$$

Sticky: $\Pr[R \geq 50] = \frac{1}{2}$

Uniform: $\Pr[R \geq 50]$ really small



How to do better for uniform?
Instead of using $\mathbb{E}[R]$, use $\mathbb{E}[R^2]$

Sticky - $\mathbb{E}[R^2] = \underbrace{\frac{100^2}{2}}_{\text{all heads}} + \underbrace{\frac{0}{2}}_{\text{all tails}} = 5{,}000$

Uniform - E_i = event i^{th} coin is heads,
$R = \sum_{i=1}^{100} \mathbb{1}_{E_i}$ $\quad (\mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \cdots)(\mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \cdots)$
$$\mathbb{E}[R^2] = \mathbb{E}\left[\left(\sum_{i=1}^{100} \mathbb{1}_{E_i}\right)^2\right] = \mathbb{E}\left[\sum_{i,j=1}^{100} \mathbb{1}_{E_i}\mathbb{1}_{E_j}\right]$$
$$= \sum_{i,j=1}^{100} \mathbb{E}[\mathbb{1}_{E_i}\mathbb{1}_{E_j}] = \sum_{i,j=1}^{100} \mathbb{E}[\mathbb{1}_{E_i \cap E_j}]$$
(linearity of expectation) $\quad (i,j) \in [100]^2$

$$= \sum_{\substack{i=j \\ i=1}}^{100} \mathbb{E}[\mathbb{1}_{E_i \cap E_j}] + \sum_{\substack{i \neq j \\ (i,j) \in [100]^2}} \mathbb{E}[\mathbb{1}_{E_i \cap E_j}]$$
$$= \sum_{i=1}^{100} \Pr[E_i] + \sum_{i \neq j} \Pr[E_i \cap E_j] \quad (E_i \text{ \& } E_j \text{ are independent})$$
$$= \sum_{i=1}^{100} \Pr[E_i] + \sum_{i \neq j} \Pr[E_i]\Pr[E_j] \quad \Pr[E_i] = \frac{1}{2}$$
$$= \sum_{i=1}^{100} \frac{1}{2} + \sum_{i \neq j} \frac{1}{4} = \frac{100}{2} + \frac{9900}{4} = 2525$$
(9900) $100 \cdot 99$ (generalized)

Chebyshev's Inequality
Thm. R is a r.v., any positive x
$$\Pr[|R - \mathbb{E}[R]| \geq x] \leq \frac{\mathbb{E}[R^2] - \mathbb{E}[R]^2}{x^2}$$

Proof. Let $Y = (R - \mathbb{E}[R])^2$
$$\Pr[|R - \mathbb{E}[R]| \geq x] = \Pr[Y \geq x^2] \overset{\text{Markov's inequality}}{\leq} \frac{\mathbb{E}[Y]}{x^2}$$
Enough to show $\mathbb{E}[Y] = \mathbb{E}[R^2] - \mathbb{E}[R]^2$
$$\mathbb{E}[Y] = \mathbb{E}[(R - \mathbb{E}[R])^2]$$
$$= \mathbb{E}[R^2 - 2\mathbb{E}[R]R + \mathbb{E}[R]^2]$$
$$= \mathbb{E}[R^2] - 2\mathbb{E}[\mathbb{E}[R]R] + \mathbb{E}[\mathbb{E}[R]^2]$$
$$= \mathbb{E}[R^2] - 2\mathbb{E}[R]\mathbb{E}[R] + \mathbb{E}[R]^2$$
$$= \mathbb{E}[R^2] - \mathbb{E}[R]^2$$
□

Uniform - $\mathbb{E}[R^2] - \mathbb{E}[R]^2 = 2525 - 50^2 = 25$
Sticky - $\mathbb{E}[R^2] - \mathbb{E}[R]^2 = 5000 - 50^2 = 2500$
$$\Pr[R \geq 75] \leq \Pr[|R - 50| \geq 25] \leq \frac{1}{25} \text{ - uniform} \quad \left(\frac{25}{25^2}\right)$$
$$\leq 4 \text{ - sticky} \quad \left(\frac{2500}{25^2}\right)$$
(if $R \geq 75$, then $|R - 50| \geq 25$)

Def. $\mathbb{E}[R^2] - \mathbb{E}[R]^2 = \mathrm{Var}[R]$, Variance
(measure of how spread out a distribution)

Thm. If R_1, R_2 are independent r.v.'s
$$\mathrm{Var}[R_1 + R_2] = \mathrm{Var}[R_1] + \mathrm{Var}[R_2]$$
Proof. $\mathrm{Var}[R_1 + R_2] = \mathbb{E}[(R_1 + R_2)^2] - (\mathbb{E}[R_1 + R_2])^2$
$$= \mathbb{E}[R_1^2] + \cancel{2\mathbb{E}[R_1 R_2]} + \mathbb{E}[R_2^2]$$
$$- \mathbb{E}[R_1]^2 - \cancel{2\mathbb{E}[R_1]\mathbb{E}[R_2]} - \mathbb{E}[R_2]^2$$
(cancel because of independence)
$$= \mathrm{Var}[R_1] + \mathrm{Var}[R_2]$$
□

Uniform dist for coins,
$$\mathrm{Var}[R] = \mathrm{Var}\left[\sum_{i=1}^{100} \mathbb{1}_{E_i}\right] = \sum_{i=1}^{100} \mathrm{Var}[\mathbb{1}_{E_i}] = \sum_{i=1}^{100} \frac{1}{4} = \frac{100}{4}$$
$\mathrm{Var}[\mathbb{1}_{E_i}] = \mathbb{E}[\mathbb{1}_{E_i}^2] - \mathbb{E}[\mathbb{1}_{E_i}]^2 = \frac{1}{2} - \frac{1}{4}$

Moment bounds / Chernoff
Thm. If q is even, $\mathbb{E}[R] = 0$, then
$$\Pr[|R| \geq x] \leq \frac{\mathbb{E}[R^q]}{x^q} \longleftarrow \text{moment}$$
$$\Pr[R \geq x] \leq \frac{\mathbb{E}[e^{tR}]}{e^{tx}} \quad t > 0$$

Proof. Moment - $Y = R^q$, $\Pr[|R| \geq x] = \Pr[Y \geq x^q] \leq \frac{\mathbb{E}[Y]}{x^q}$

Chernoff - $Y = e^{tR}$, $\Pr[R \geq x] = \Pr[Y \geq e^{tx}] \leq \frac{\mathbb{E}[Y]}{e^{tx}}$