May 5

Wednesday, May 5, 2021 2:21 PM

Comp Sci 212

1. Conditional Expectation

2. Expectation of Product

Announcements

- Midterm grades out
- HW4 out

Binomial distribution

$S = \{0,1\}^n$, $\Pr[\omega] = p^{\#\text{ of 1's in } \omega}(1-p)^{\#\text{ of 0s in } \omega}$

$E_i = \{\omega \mid \omega \in S,\ \omega_i = 1\}$ $\quad 0 \le p \le 1$

$$\Pr[E_i] = \sum_{\substack{\omega \in E_i \\ \omega \in \{0,1\}^n \\ \text{s.t. } \omega_i = 1}} p^{\#\text{ of 1's } \omega}(1-p)^{\#\text{ of 0's in } \omega}$$

$$p \cdot \sum_{\substack{\omega \in \{0,1\}^n \\ \text{s.t. } \omega_i = 1}} p^{\#\text{ of 1's in } \omega - 1}(1-p)^{\#\text{ of 0's in } \omega}$$

$$p \cdot \underbrace{\sum_{\omega \in \{0,1\}^{n-1}} p^{\#\text{ of 1's in } \omega}(1-p)^{\#\text{ of 0s in } \omega}}_{=1\ \text{binomial dist. } n-1}$$

$$= p$$

Def. Conditional Expection, X r.v. (random variable) A event $A \subseteq S$

$$\mathbb{E}[X \mid A] = \sum_{\omega \in A} X(\omega) \cdot \Pr[\omega \mid A]$$

$$= \frac{1}{\Pr[A]} \sum_{\omega \in A} X(\omega) \cdot \Pr[\omega]$$

Law of total expectation. R random variable, $A_1, A_2, \ldots, A_n$ disjoint, $A_1 \cup A_2 \cup \cdots \cup A_n = S$

$$\mathbb{E}[R] = \sum_{i=1}^{n} \mathbb{E}[R \mid A_i] \cdot \Pr[A_i].$$

Proof. $$\sum_{i=1}^{n} \mathbb{E}[R \mid A_i] \Pr[A_i] = \sum_{i=1}^{n} \sum_{\omega \in A_i} X(\omega) \cdot \Pr[\omega]$$

$$= \sum_{\omega \in S} X(\omega) \Pr[\omega] = \mathbb{E}[R]$$

$$\mathbb{E}[R \mid A_2] = \frac{1}{1-p} \cdot \sum_{i=2}^{\infty} i \cdot (1-p)^{i-1} \cdot p$$

$$= \sum_{i=2}^{\infty} i \cdot (1-p)^{i-2} \cdot p = \sum_{i=1}^{\infty} (i+1)(1-p)^{i-1} \cdot p$$

$$= \underbrace{\sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p}_{\mathbb{E}[R]} + \underbrace{\sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p}_{=1}$$

Mean time until failure

Each day your laptop fails w.p. (with probability) p, how long until it fails on average? (expected value).

$S = \{1, 2, 3, \ldots\}$, $0 < p \le 1$ → geometric distribution

$\Pr[i] = (1-p)^{i-1} \cdot p$ $\longrightarrow$ $\sum_{i=1}^{\infty} p \cdot (1-p)^{i-1} =$

$i \in S$

$$p \underbrace{\sum_{i=1}^{\infty} (1-p)^{i-1} = p \cdot \frac{1}{1-(1-p)}}_{\text{sum of geometric sequence}} = 1$$

$R: S \to S$, $R(i) = i$

$\mathbb{E}[R] = \sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p \cdot R(i)$ → can compute

Let $A_1 = \{1\}$, $A_2 = \{2, 3, \ldots\}$ $A_1 \cup A_2 = S$

$\Pr[A_1] = p$, $\Pr[A_2] = 1-p$

$$\mathbb{E}[R] = \mathbb{E}[R \mid A_1] \cdot \Pr[A_1] + \mathbb{E}[R \mid A_2]\Pr[A_2]$$

$$= 1 \cdot p + (1-p)(1 + \mathbb{E}[R])$$

$$= 1 + (1-p)\mathbb{E}[R], \text{ solve } \boxed{\mathbb{E}[R] = \frac{1}{p}}$$

$\mathbb{E}[R] = 1 + (1-p)\mathbb{E}[R]$

$p \cdot \mathbb{E}[R] = 1$, $\mathbb{E}[R] = 1/p$

Fact. $$\mathbb{E}[X] = \sum_{r \in \text{Range}(X)} r \cdot \Pr[X = r]$$

all the values X takes on

Proof. $$\mathbb{E}[X] = \sum_{\omega \in S} X(\omega) \Pr[\omega]$$

$$= \sum_{r \in \text{Range}(X)} \sum_{\omega: X(\omega) = r} \Pr[\omega] \cdot \overbrace{X(\omega)}^{=r}$$

$$= \sum_{r \in \text{Range}(X)} r \sum_{\omega: X(\omega) = r} \Pr[\omega]$$

$$= \sum_{r \in \text{Range}(X)} \Pr[X = r] \cdot r$$

$\square$

Expected Value of Product

Thm. If R_1 & R_2 are independent r.v.'s,

$$\mathbb{E}[R_1 \cdot R_2] = \mathbb{E}[R_1] \cdot \mathbb{E}[R_2]$$

$\Pr[R_1 = x_1 \text{ and } R_2 = x_2] = \Pr[R_1 = x_1] \cdot \Pr[R_2 = x_2]$

$(R_1 \cdot R_2(\omega) = R_1(\omega) \cdot R_2(\omega))$

Proof. $$\mathbb{E}[R_1 \cdot R_2] = \sum_{r \in \text{Range}(R_1 \cdot R_2)} \Pr[R_1 \cdot R_2 = r] \cdot r$$

$$= \sum_{\substack{r_1 \in \text{Range}(R_1) \\ r_2 \in \text{Range}(R_2)}} \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \cdot r_1 \cdot r_2$$

$$= \sum_{\substack{r_1 \in \text{Range}(R_1) \\ r_2 \in \text{Range}(R_2)}} \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2] \cdot r_1 \cdot r_2 \quad \text{(independence)}$$

$$= \left(\sum_{r_1 \in \text{Range}(R_1)} \Pr[R_1 = r_1] \cdot r_1\right)\left(\sum_{r_2 \in \text{Range}(R_2)} \Pr[R_2 = r_2] \cdot r_2\right)$$

$$= \mathbb{E}[R_1] \cdot \mathbb{E}[R_2].$$

$\square$

Cauchy - Schwartz

Thm. If R_1, R_2, not necessarily independent

$$\mathbb{E}[R_1 \cdot R_2] \le \mathbb{E}[R_1^2]^{1/2} \cdot \mathbb{E}[R_2^2]^{1/2}$$