Linearity of Expectation

Thm. R_1, R_2, random variables

R_1+R_2, $R_1+R_2(\omega) = R_1(\omega)+R_2(\omega)$

$\mathbb{E}[R_1+R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$

Proof. $\mathbb{E}[R_1+R_2] = \sum_{\omega \in S} R_1+R_2(\omega)\Pr[\omega]$

$= \sum_{\omega \in S} R_1(\omega)\Pr[\omega] + R_2(\omega)\Pr[\omega]$

$= \sum_{\omega \in S} R_1(\omega)\Pr[\omega] + \sum_{\omega \in S} R_2(\omega)\Pr[\omega]$

$= \mathbb{E}[R_1] + \mathbb{E}[R_2]$ $\square$

Coins - $S = \{H,T\}^{100}$ (uniform & sticky coin)

$R: S \to \mathbb{R}$ $R(s)$ = # of heads in s

$\mathbf{1}_{E_i}$, $E_i = \{s \mid s \in S, s_i = H\}$

event i^{th} coin is heads.

$R(s) = \sum_{i=1}^{100} \mathbf{1}_{E_i}(s)$

$\mathbb{E}[R] = \mathbb{E}[\sum_{i=1}^{100} \mathbf{1}_{E_i}] \overset{\text{linearity of expectation}}{=} \sum_{i=1}^{100} \mathbb{E}[\mathbf{1}_{E_i}]$

$= \sum_{i=1}^{100} \Pr[E_i] = \sum_{i=1}^{100} \frac{1}{2} = 50$

indicator random variables

generally, let $E_1, \dots, E_n$ be events

$R: S \to \mathbb{R}$, $R(s)$ = # of events s is in

then $\mathbb{E}[R] = \sum_{i=1}^{n} \Pr[E_i]$

(*)

Binomial distribution

$S = \{0,1\}^n$ $\Pr[\omega] = p^{\#\text{ of 1's in }\omega} \cdot (1-p)^{\#\text{ of 0's in }\omega}$

$0 \leq p \leq 1 \to$ parameter

$0 \leq \Pr[\omega]$

$\sum_{\omega \in S} \Pr[\omega] = \sum_{\omega \in \{0,1\}^n} p^{\#\text{ of 1's in }\omega} (1-p)^{\#\text{ of 0's in }\omega} = \sum_{i=0}^{n} \sum_{\substack{\omega \text{ s.t.} \\ \omega \text{ has } i \text{ 1's} \\ \omega \in \{0,1\}^n}} p^i \cdot (1-p)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \cdot p^i (1-p)^{n-i}$

$= (p+(1-p))^n = 1$

binomial theorem

$E_i = \{\omega \mid \omega \in S, \omega_i = 1\}$, strings i^{th} element is 1, $\Pr[E_i] = p$ → prove on Wednesday

(prove later)

$R: S \to \mathbb{R}$, $R(s)$ = # of 1's in s, $R(s) = \sum_{i=1}^{n} \mathbf{1}_{E_i}(s)$

$\mathbb{E}[R] = \underbrace{\sum_{\omega \in S} \Pr[\omega] \cdot (\#\text{ of 1's in }\omega)}_{\text{def. of } \mathbb{E}} = \sum_{i=0}^{n} \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i} \cdot i = pn$

$= \sum_{i=1}^{n} \Pr[E_i] = \sum_{i=1}^{n} p = p \cdot n$

(Using *)