April 26

Monday, April 26, 2021 1:36 PM

Comp Sci
1. Law of Total Probability
2. Independence
3. Pairwise Independence

Announcements
- Midterm May 3 - In class, Crowdmark
- Practice Midterms up

Conditional Probability

$$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$$

Law of total probability

Thm. If $E_1, \ldots, E_n$ are disjoint,

$$Pr[A \cap (E_1 \cup E_2 \cup \cdots \cup E_n)] = \sum_{i=1}^{n} Pr[A \mid E_i] \cdot Pr[E_i]$$

Union of E_i's make operative space.

Proof. $\sum_{i=1}^{n} Pr[A \mid E_i] \cdot Pr[E_i] = \sum_{i=1}^{n} Pr[A \cap E_i] = \sum_{i=1}^{n} \sum_{w \in A \cap E_i} Pr[w]$

because E_i are disjoint, each w appears in only one of the $A \cap E_i$

$= \sum_{w \in A \cap (E_1 \cup E_2 \cup \cdots E_n)} Pr[w] = Pr[A \cap (E_1 \cup \cdots \cup E_n)]$

Ex. Roll two dice, what is the probability that the sum is 4?

$S = [6]^2$, uniform dist. $E_1 \cup E_2 \cup \cdots \cup E_6 = S$

E_i = event that the first die is i → all disjoint

$= \{w \mid w \in S, w_1 = i\}$

A = event that the sum is 4

$= \{w \mid w \in S, w_1 + w_2 = 4\}$

$|E_i| = 6$

$Pr[A \mid E_i] = \begin{cases} \frac{1}{6} & i \le 3 \\ 0 & i \ge 4 \end{cases}$ → $Pr[E_i] = \frac{1}{6}$, $Pr[A \cap E_i] = \frac{1}{|S|} = \frac{1}{36}$

$Pr[A] = Pr[A \cap (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6)]$

$= \sum_{i=1}^{6} Pr[A \mid E_i] \cdot Pr[E_i]$

$= \sum_{i=1}^{3} \frac{1}{6} \cdot \frac{1}{6} + \sum_{i=4}^{6} 0$

$= \frac{3}{36} = \frac{1}{12}$.

Independence

Events A + B are independent if

$Pr[A \mid B] = Pr[A]$ - restricting to B doesn't affect $Pr[A]$

$Pr[A \cap B] = Pr[A] \cdot Pr[B]$

↳ include $Pr[B] = 0$

$E_1, E_2, \ldots, E_n$ mutually independent if for every $T \subseteq [n]$, - Independence is something

$$\underbrace{Pr[\bigcap_{j \in T} E_j]}_{\text{intersection of } E_j\text{'s}} = \underbrace{\prod_{j \in T} Pr[E_j]}_{\text{product of } Pr[E_j] \text{ for all } j \in T}$$

$S = \{H, T\}^{100}$, uniform

Coins $E_i = \{w \mid w \in S, w_i = H\}$

i^{th} coin is heads.

Uniform

$T \subseteq [100]$

$$Pr[\bigcap_{j \in T} E_j] = \frac{|\{w \mid w_j = H \text{ for } j \in T\}|}{2^{100}}$$

$= \frac{2^{100-|T|} \cdot 1^{|T|}}{2^{100}}$ generalized product rule → 2 choices for coins not in T - $\#100-|T|$; 1 coins in T

$= \frac{1}{2^{|T|}}$

$= \prod_{j \in T} \frac{1}{2} = \prod_{j \in T} Pr[E_j]$

so they're mutually independent

Sticky coins - $Pr[\text{all heads}] = Pr[\text{all tails}] = \frac{1}{2}$ $(H, H, H, \ldots)$

$Pr[\bigcap_{j \in T} E_j] = \frac{1}{2} \ne \prod_{j \in T} Pr[E_j] = \frac{1}{2^{|T|}}$

$\frac{1}{2}$

not independent.

Pairwise Independence

$E_1, E_2, \ldots, E_n$ are pairwise independent if for every i, j, E_i + E_j are independent.

Ex. $S = \{H, T\}^2$ uniform dist.

$E_1 = 1^{st}$ coin is heads $= \{(H, T), (H, H)\}$ $Pr[E_i] = \frac{1}{2}$

$E_2 = 2^{nd}$ coin heads $\{(H, H), (T, H)\}$ $Pr[E_1 \cap E_2] = \frac{1}{4}$

E_3 = both coins same $\{(H, H), (T, T)\}$ $Pr[E_1 \cap E_3] = \frac{1}{4}$ $Pr[E_2 \cap E_3] = \frac{1}{4}$

$Pr[E_1 \cap E_2 \cap E_3] = \frac{1}{4} \ne Pr[E_1] \cdot Pr[E_2] \cdot Pr[E_3]$

pairwise independent, not mutually independent.

$S = \{H, T\}^n$ uniform dist. n coins - mutual independent events

$R \subseteq [n]$, $R \ne \emptyset$ $2^n - 1$ pairwise independent events

$E_R = \{w \mid w \in S, \#. i \text{ s.t. } i \in R, w_i = H \text{ is odd}\}$

$Pr[E_R] = \frac{1}{2}$

$Pr[E_{R_1} \cap E_{R_2}] = \frac{1}{4}$

↳ $\frac{|E_R|}{2^n} = \frac{2^{n-|R|} \left(\binom{|R|}{1} + \binom{|R|}{3} + \binom{|R|}{5} + \cdots \right)}{2^n}$

↳ $R_1 \setminus R_2$ $R_2 \setminus R_1$ $R_1 \cap R_2$

$R_1 \setminus R_2$	$R_2 \setminus R_1$	$R_1 \cap R_2$
even	even	odd
odd	odd	even