April 21

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Comp Sci 212

1. Pigeonhole Principle
2. More bijection rule

Announcements

- Midterm Plan
- Getting to know you meetings

Pigeonhole principle

If number of pigeons is greater than the number of holes, then there is a hole w/ more than 1 pigeon.

If $f: A \to B$, is a total function, $k|B| < |A|$ then there exists $b \in B$ s.t. $|f^{-1}(b)| \geq k+1$.

Theorem. In a sock drawer you have black + blue socks. If you pick 3, you have pair w/ same color.

Proof. Let the socks be $S = \{s_1, s_2, s_3\}$, colors $C = \{\text{black}, \text{blue}\}$

$f: S \to C$ $f(s) =$ color of s. $|C| < |S|$, exist $b \in B$ s.t. $|f^{-1}(b)| \geq 2$ by pigeonhole principle.

I.e. there is a pair w/ same color. □

Theorem. Let $S \subseteq \underbrace{[20]}_{\{1,2,\ldots,20\}}$ s.t. $|S| \geq 11$. Then there exist $i, j \in S$ s.t. $i - j = 10$.

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Possible pairs – (1, 11) (2, 12) (3, 13) (4, 14) (5, 15) (6, 16) (7, 17) (8, 18) (9, 19) (10, 20)

Proof. Let $f: S \to [10]$ defined by

$f(x) = \begin{cases} x - 10 & \text{if } x > 10 \\ x & \text{o.w.} \end{cases}$ otherwise

$|S| > |[10]| = 10$

By pigeonhole principle, if $|S| \geq 11$, there exist $b \in [10]$ s.t. $|f^{-1}(b)| \geq 2$. These two elements have a difference of 10. □

$f(14) = 4$, $f(4) = 4$

Thm. $\binom{n}{k} = \binom{n}{n-k}$

Proof. $|A| = \binom{n}{k} =$ # of subsets of $[n]$ of size k

$A = \{x \mid x \subseteq [n], |x| = k\}$

$|B| = \binom{n}{n-k} =$ # of subsets of $[n]$ of size $n-k$

$B = \{x \mid x \subseteq [n], |x| = n-k\}$

$f: A \to B$ $n = 6, k = 2$, $f(\{1,5\}) = \{2,3,4,6\}$

$f(a) = [n] \setminus a$

set of elements in $[n]$ but not in a

$f^{-1}(b) = \{[n] \setminus b\}$, $|f^{-1}(b)| = 1$, so f is bijection

by bijection rule $|A| = |B|$ □

If $f(a) = b$

a is a subset of $[n]$

every element not in b, but in $[n]$ is in a

every element in b is not in a

Thm. $\underbrace{\binom{n}{k}}_{|A|} = \sum_{i=0}^{k} \binom{n/2}{i}\binom{n/2}{k-i}$ multiplication, assume n is even $= |D|$

Proof. A = set of subsets of $[n]$ of size k same as before

$B_i = \{x \mid x \subseteq [n/2], |x| = i\}$

$C_i = \{x \mid x \subseteq [n] \setminus [n/2],$

$[n] \setminus [n/2] = \{n/2+1, n/2+2, \ldots, n$

$|B_i| = \binom{n/2}{i}$

$|C_i| = \binom{n/2}{k-i}$

$\sum_{i=0}^{k} |B_i| \cdot |C_i| = \sum_{i=0}^{k} \binom{n/2}{i}\binom{n/2}{k-i}$

product rule

$= \sum_{i=0}^{k} |B_i \times C_i| \underset{\text{sum rule}}{=} \left| \bigcup_{i=0}^{k} B_i \times C_i \right| = D$

union from i to k,

$B_0 \times C_0 \cup B_1 \times C_1 \cup B_2 \times C_2 \cdots$

$f: A \to D$ if f is a bijection, then theorem is proved.

$f(a) = (a \cap [n/2], a \cap ([n] \setminus [n/2]))$

$n = 10, k = 4$

$a = \{1, 2, 3, 7\}$ $a = \{4, 5, 6, 7)$

$f(a) = (\{1,2,3\}, \{7\})$ $f(a) = (\{4,5\}, \{6,7\})$.

$d \in D$ $d = (b, c) \in B_i \times C_i$ for some i

$f^{-1}((b,c)) = \{b \cup c\} \to$ If $f(a) = (b, c)$

Therefore, $|f^{-1}((b,c))| = 1$ for all $(b,c) \in D$, f is a bijection.

$b \subseteq a$, $c \subseteq a$, so $b \cup c \subseteq a$

$|a| = k$

$a = b \cup c$

$|b \cup c| \underset{\text{sum rule}}{=} |b| + |c| = i + k - i = k$

If $A \subseteq B$, $|A| = |B|$, then $A = B$ □

finite