April 23

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Comp Sci 212

1. Probability
2. Conditional Probability

Announcements

- HW1 Solutions out

Probability

Def. Sample space - non-empty set S

$w \in S$ - outcomes

subset of S - event

Ex. $S = \{1,2,3,4,5,6\}$

$S = \{\text{Heads}, \text{Tails}\}$

Def. Probability function - total function

$Pr: S \to \mathbb{R}$, s.t.

$Pr[w] \geq 0$ for every $w \in S$

$\underbrace{\sum_{w \in S}}_{\text{sum over all } w \in S} Pr[w] = 1$

Ex.

w	$Pr[w]$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

w	$Pr[w]$
1	$\frac{1}{2}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{10}$

If $\underbrace{E}_{\text{event}} \subseteq S$ $Pr[E] = \sum_{w \in E} Pr[w]$

Ex. $E = \{x \mid x \in [6], x \leq 3\}$

$Pr[E] = Pr[1] + Pr[2] + Pr[3] = Pr[X \leq 3]$

Ex Uniform distribution $Pr[w] = \frac{1}{|S|}$ for all w.

$Pr[E] = \frac{|E|}{|S|}$

Let $S = \{H, T\}^{100} \to$ flipping a coin 100 times

Uniform distribution, $|S| = 2^{100}$, $Pr[w] = \frac{1}{2^{100}}$

$Pr[\text{at least 75 heads}] = Pr[E] = \frac{|E|}{2^{100}} \to$ can solve using combinatorics

$E = \{s \mid s \in S, \text{ at least 75 H's in } s\}$

Sticky coin

$Pr[\text{All heads}] = \frac{1}{2}$ not uniform

$Pr[\text{All tails}] = \frac{1}{2}$

$Pr[\text{everything else}] = 0$

$Pr[\text{at least 75 heads}] = \frac{1}{2}$

Probability rules

- Sum rule, if $E_1, \ldots, E_n$ are disjoint,

$Pr[\underbrace{\bigcup_{i=1}^{n} E_i}_{\text{union of } E_i,\ E_1 \cup E_2 \cup \cdots \cup E_n}] = \sum_{i=1}^{n} Pr[E_i]$

- Complement rule
  - $Pr[\underbrace{S \setminus E}_{\text{set of elements in } S \text{ but not } E}] = 1 - Pr[E]$
  - Difference rule
    - $Pr[E_1 \setminus E_2] = Pr[E_1] - Pr[E_1 \cap E_2]$
  - If $E_1 \subseteq E_2$, $Pr[E_1] \leq Pr[E_2]$
  - Inclusion-Exclusion

$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$

- Union bound
  - $Pr[A \cup B] \leq Pr[A] + Pr[B] \to$
  - $Pr[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} Pr[A_i]$

Proof. $Pr[A \cup B] = \sum_{w \in A \cup B} Pr[w]$

$\leq \sum_{w \in A} Pr[w] + \sum_{w \in B} Pr[w] = Pr[A] + Pr[B]$

w is in A but not B, or in B but not A, it appears in 1 sum,

w is in both, it appears in two sum, $Pr[w] \geq 0$, it only makes right-hand side bigger □

Ex. 100-sided die, throw it 25 times

$S = [100]^{25}$, uniform distribution

$Pr[\text{at least 1 roll is a 1}] = Pr[A_1 \cup A_2 \cup \cdots \cup A_{25}]$

A_i = event that the i^{th} roll is 1

$= \{w \mid w \in S, w_i = 1\}$

$\leq \sum_{i=1}^{25} Pr[A_i]$

$= \sum_{i=1}^{25} \frac{|A_i|}{|S|}$

$= \sum_{i=1}^{25} \underbrace{\frac{100^{24}}{100^{25}}}_{\text{product rule}} = \sum_{i=1}^{25} \frac{1}{100} = \frac{25}{100}$

$|S^k| = |S|^k$

$|A_i| = |\{1\} \times \underbrace{[100] \times [100] \times \cdots \times [100]}_{24 \text{ times}}| = 100^{24}$

Given that a person has two kids, one is a boy, what is the probability both are boys?

$S = \{(b,b), (b,g), (g,b), (g,g)\}$, uniform

$B = \{(b,g), (g,b), (b,b)\}$

$A = \{(b,b)\}$

$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$

$Pr[A] = \frac{1}{4}$ $= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

$A \cap B = A$

$Pr[A \cap B] = Pr[A]$

Conditional probability -

What happens if we restrict sample space?

$S = [6]^2$, uniform distribution (die rolls) → two elements of pair

Given that the sum is 4, what is the probability that 1^{st} die roll is 1?

$\to Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$

conditioning on B

$A \cap B = \{(1,3)\}$, $|S| = 36$

B = set of outcomes whose sum is 4

$= \{(3,1), (2,2), (1,3)\}$

A = set outcomes 1^{st} roll is 1

$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$