April 19

Monday, April 19, 2021 1:47 PM

Comp Sci 212

1. Binomial Theorem

2. Principle of Inclusion-Exclusion

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = # of subsets of $[n]$ of size k

"n choose k"

Theorem. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Proof. Consider $[n+1] = \{1, 2, 3, \ldots, n+1\}$

Let A be the set of subsets of $[n+1]$ of size k

Let A_1 be the set of subsets of $[n]$ of size k

Let A_2 be the set of subsets of $[n]$ of size $k-1$.

Need to prove $|A| = |A_1| + |A_2|$.

A_1 & A_2 are disjoint, so $|A_1| + |A_2| = |A_1 \cup A_2|$ by sum rule.

$f: A \to A_1 \cup A_2$

$f(x) = \begin{cases} x & \text{if } x \text{ doesn't contain } n+1 \\ x \setminus \{n+1\} & \text{if } x \text{ does contain } n+1 \end{cases}$

x w/o the element $n+1$

$n=6$, $k=3$

$f(\{1,4,7\}) = \{1,4\}$

$f(\{1,4,6\}) = \{1,4,6\}$

$f^{-1}(y) = \begin{cases} \{y\} & \text{if } |y| = k \\ \{y \cup \{n+1\}\} & \text{if } |y| = k-1 \end{cases}$, $|f^{-1}(y)| = 1$ for all y

f is a bijection, so $|A| = |A_1 \cup A_2| = |A_1| + |A_2|$. □

To choose k elements from $[n+1]$

choose k elements from $[n]$ $k-1$ elements from $[n]$, and $n+1$

$\binom{n}{k} = 0$ if $k \leq -1$ or $k \geq n+1$ ($=1$ if $k=0$)

Binomial Theorem.

Thm. If n is non-negative integer,

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^i = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \cdots + \binom{n}{n}x^n$$

Proof. Use induction.

Base case - $n=0$, $1 = \binom{n}{0} \cdot x^0 = 1$

Inductive step - Assume $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$

Then $(1+x)^{n+1} = (1+x)(1+x)^n$

$= (1+x)\left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)$

$= \sum_{i=0}^{n} \binom{n}{i} x^i + \sum_{i=0}^{n} \binom{n}{i} x^{i+1}$ $\quad \binom{n}{0}x + \binom{n}{1}x^2 \ldots$

$= \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right) + \left(\sum_{i=1}^{n+1} \binom{n}{i-1} x^i\right) + \underbrace{\binom{n}{n+1}}_{=0} x^{n+1} + \underbrace{\binom{n}{-1}}_{=0} x^0$

$= \sum_{i=0}^{n+1} \binom{n}{i} x^i + \sum_{i=0}^{n+1} \binom{n}{i-1} x^i$

$= \sum_{i=0}^{n+1} x^i \left(\binom{n}{i} + \binom{n}{i-1}\right)$

$= \sum_{i=0}^{n+1} \binom{n+1}{i} x^i$ (by 1st thm). □

Corollary. If $n \in \mathbb{N}$, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Proof. $\left(1 + \frac{a}{b}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{a^i}{b^i}$ by binomial thm.

Thus, $(a+b)^n = b^n \left(1 + \frac{a}{b}\right)^n$

$= b^n \sum_{i=0}^{n} \binom{n}{i} \frac{a^i}{b^i}$

$= \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ □

Theorem. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$

Proof. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i} \cdot 1^i$

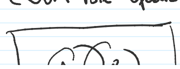
(binomial theorem) $= (1+1)^n$

$= 2^n$ □

Principle of Inclusion-Exclusion

For all A, B, $|A \cup B| = |A| + |B| - |A \cap B|$

(Sum rule update)



(binomial theorem) $= (1+1)^n$

$= 2^n$ $\square$

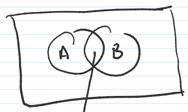
Theorem. $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$

Proof. $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}\cdots = \sum_{i=0}^{n}\binom{n}{i}\cdot(-1)^i$

(by binomial theorem) $= (1-1)^n$

$= 0$ $\square$

(Sum rule update)



$A \cap B$ - gets double-counted

A, B, C $\quad |A\cup B\cup C| = |A|+|B|+|C| - |A\cap B|$
$- |A\cap C| - |B\cap C|$
$+ |A\cap B\cap C|$

Ex. How many #'s from 1 to 100 are divisible by 2 or 5?

$A = \{x \mid 1 \le x \le 100,\ x \text{ divisible by } 2\}$ $\quad |A| = 50$

$B = \{x \mid 1 \le x \le 100,\ x \text{ divisible by } 5\}$ $\quad |B| = 20$

$A\cap B = \{x \mid 1 \le x \le 100,\ x \text{ divisible by } 2 \,\&\, 5\}$ $\quad |A\cap B| = 10$

$\underset{10}{}$

$|A\cup B| = |A|+|B|-|A\cap B|$
$= 50+20-10 = 60.$