April 16

Friday, April 16, 2021 2:05 PM

Comp Sci 212

1. Counting Rules etc.

Generalized Product Rule. If S is a set of length-k sequences, $s \in S$, $s = (s_1, s_2, \dots, s_k)$

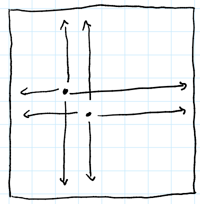
- n_1 = # of possibilities for s_1
- n_2 = # of possibilities for s_2 given s_1
- n_3 = # of possibilities for s_3 given s_1, s_2

$\vdots$

- n_k = # of possibilities for s_k given $s_1, \dots, s_{k-1}$

$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$

Ex. How many ways are there to place k labeled rooks on a $k \times k$ chessboard?
- every row to have at most 1 rook
- every column to have at most 1 rook



Use generalized product rule

$n_1 = k^2$

$n_2 = (k-1)^2$

$n_3 = (k-2)^2$

$\vdots$

$n_i = (k-i+1)^2$

$\vdots$

$n_k = 1$

Total number $= n_1 \cdot n_2 \cdots n_k$

$= k^2 (k-1)^2 \cdots 1$

$= (k \cdot (k-1) \cdots 1)^2$

$= (k!)^2$

factorial: $k! = k \cdot (k-1) \cdots 1$

Def. Permutation - given a set of size k, is a length-k sequence of elements from that set, each element appears exactly one

$S = \{a, b, c, d\}$ permutations (c, b, a, d), (a, b, c, d)

Thm. # of permutations of S, $|S| = k$, is $k!$

Proof. Use generalized product rule

$n_1 = k$, because first element can be anything

$n_2 = k-1$, given s_1, s_2 can be anything else

$n_i = k-i+1$

$\vdots$

$n_k = 1$

Total # $= k \cdot (k-1) \cdots 1$
$= k!$

□

Sorting: n permutation $\rightarrow$ sorted list

program - input permutation
applies swaps
output should be a sorted list

Merge-sort $O(n \log n)$ running time

Can we do better? No

A = set of permutations

B = set of sequences of swaps encode in $\{0,1\}^\ell$ ← ℓ time it takes to apply swaps

program $f: A \rightarrow B$, $C = \{b \mid f(a) = b\}$

$f(a) = f(b)$ implies $a = b$

$|A| = |C| = n! \leq |B| = 2^\ell$

$\rightarrow \log(n!) \leq \ell$

$\log(1) + \log(2) + \cdots + \log(n) = \Theta(n \log n)$

$\geq \underbrace{\log(\tfrac{n}{2}) + \cdots + \log(\tfrac{n}{2})}_{n/2 \text{ times}}$

Division rule

Def. $f: A \rightarrow B$ is k-to-1 if $|f^{-1}(b)| = k$ for every $b \in B$. $f^{-1}(b) = \{a \mid f(a) = b\}$

Division rule - If $f: A \rightarrow B$ is k-to-1, $|A| = k \cdot |B|$.

Thm. Set S, $|S| = k$, # length-ℓ sequences of elements from S, no two the same $= \dfrac{k!}{(k-\ell)!}$

Proof. A = set of permutations of S.
B = set of length-ℓ sequences.

$f: A \rightarrow B$, $f(a)$ = first ℓ elements of a. → $k=6$, $\ell=2$, $a = (6,1,2,3,5,4)$, $f(a) = (6,1)$

Need to prove $|f^{-1}(b)| = (k-\ell)!$ → $|A| = (k-\ell)! \cdot |B|$ by division rule, or $|B| = \dfrac{k!}{(k-\ell)!}$

$f^{-1}(b) = \{$permutations whose first ℓ elements agree w/ $b\}$ → $|f^{-1}(b)| = (k-\ell)!$

b, some permutation of elements in S, but not b → $k-\ell$

Thm. # of subsets of size ℓ of S s.t. $|S| = k$, is $\dfrac{k!}{\ell!(k-\ell)!} = \binom{k}{\ell}$ - "k choose ℓ"

Proof. B = set of length-ℓ sequences of S no two the same

C = set of size-ℓ subsets of S

$f: B \rightarrow C$, $f(a) = a$, viewed as a set

$k=6$, $\ell=2$, $f((6,1)) = \{6,1\}$

$f^{-1}(c)$ = set of permutations of c

$|f^{-1}(c)| = \ell!$ for all $c \in C$

→ by division rule $|B| = \ell! \, |C|$, so $|C| = \dfrac{k!}{\ell!(k-\ell)!}$

Thm. # of sequences of length 10 w/ 8 0's, 2 1's
= # of subsets of $[10]$ of size 2 $= \binom{10}{2}$

Proof. B = set of sequences w/ 8 0's, 2 1's

C = set / collection of subsets of $[10]$ of size 2.

$f: B \rightarrow C$, $f(b) = \{i \mid b_i = 1\}$.

If $c \in C$, $c = \{i, j\}$

$f^{-1}(c) = \{(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})\}$

$b_i = b_j = 1$, rest are 0.

f is a bijection, so $|B| = |C|$.

□

Can generalize

ways to buy n donuts of k types =

of strings length $n+k-1$, $k-1$ 1's =

# of subsets of $[n+k-1]$ of size $k-1$ =

$\binom{n+k-1}{k-1}$