April 7

Wednesday, April 7, 2021 1:47 PM

Comp Sci 212

1. Strong Induction
2. Invariants

Announcements

- HW1 out on Crowdmark
  - TA / Office hours start this week
  - Discussion - HW-like problems
- Reading for 4/9
- Update to April 5 notes

Strong Induction

- Prove $P(0)$ - Base case
- Prove $P(0), \dots, P(n)$ imply $P(n+1)$ - Inductive step

McDonalds sells chicken mcnuggets in packs of 3 and 7. For what n can you buy exactly n chicken mcnuggets?

Thm. For every integer n $n \geq 12$, you can buy exactly n chicken mcnuggets.

$n = 12$ - 3, 3, 3, 3 $n = 15$ - 3, 3, 3, 3, 3

$n = 13$ - 3, 3, 7 $n = 16$ - 3, 3, 3, 7

$n = 14$ - 7, 7 $n = 17$ - 3, 7, 7

Proof. Use strong induction.

Base case, $n = 12$ can do 3, 3, 3, 3

$n = 13$ 3, 3, 7

$n = 14$ 7, 7

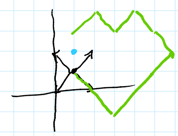
Inductive step - Assume you can buy any amount of chicken mcnuggets up to & including $n-1$. Starting from 12. To buy n chicken mcnuggets, buy $n-3$, and a pack of 3. Because $n \geq 15$, $n-3$ chicken mcnuggets is covered in base case.

Invariant - property that stays the same over time

Setting - bishop on grid starting at $(1, 1)$

can only move diagonally.

Can the bishop get to $(1, 2)$



Thm. Let (x_t, y_t) be position at time t.

Then $\underbrace{x_t + y_t \text{ is always even}}_{\text{invariant}}$.

$(1,1) \to (0,2) \to (1,3) \to (2,4) \to (3,3)$

2 2 4 6 6

Proof. Use induction.

Base case - $x_1 = 1$, $y_1 = 1$, $x_1 + y_1 = 2$ even.

Inductive step - Assume $x_t + y_t$ is even.

Use cases Then,

Case 1 - $x_{t+1} = x_t + 1$, $y_{t+1} = y_t + 1$, $x_{t+1} + y_{t+1} = \underline{x_t + y_t + 2}$ also even.

Case 2 - $x_{t+1} = x_t + 1$, $y_{t+1} = y_t - 1$

$x_{t+1} + y_{t+1} = x_t + y_t$ $x_{t+1} + y_{t+1}$ stay even

Case 3 - $x_{t+1} = x_t - 1$, $y_{t+1} = y_t + 1$

Case 4 - $x_{t+1} = x_t - 1$, $y_{t+1} = y_t - 1$

$x_{t+1} + y_{t+1} = x_t + y_t - 2$, also even $\square$

Corollary (→ Proposition) - $(1, 2)$ can not be reached.

Proof - $1 + 2 = 3$ is odd.

General Strategy - Find a property

$P(t)$ is true if the property holds at time t.

Show $P(t)$ is true for all t using induction.

Number slider puzzle

n_0

1	2	3
4	5	6
7	8	

Figure 1

Is it possible

1	2	3
4	5	6
8	7	

Figure 2

Invariant -

2	1	4
5	8	6
3		7

Write the list of numbers

$(2, 1, 4, 5, 8, 6, 3, 7) \to$ (2,1) (8,3) (8,7) (4,3) (6,3) (5,3) (8,6)

n_t = # of pairs that are out of order.

Thm. Let n_t be defined as before. Then n_t is even. (if it's reachable from starting).

(Sketch)

Prof. Use induction. In base case $t = 0$, everything is in order so $n_0 = 0$.

1	2	3
4	5	6
7	8 →	↓

Row move - move within a row → keeps the order same.

Column move - move within a column

→ move up

$\underline{\text{move down}}$ → ~~, a, b, c ~~

~~, b, c, a ~~

| b c $\overset{a}{\downarrow}$ |

(a, b) → (b, a)

$\underline{(a, c)}$ $\underline{(c, a)}$

exactly 1 out order

exactly 1 out order

change 2, 0, -2

In computer science loops,

array / list A (Insertion sort)

for i from 1 to A.length

do something

end

Invariant - proof that your loop works

after time i, subarray from 1 to i has a property → sorted

after loop runs, entire array has property.

Corollary - Figure 2 can not be reached from Figure 1.

Proof. Figure 2 has 1 pair out of order.

General n column move affect n-1 pairs

n odd, works

n=4 +3, +1, -1, -3

n=6 5, 3, 1, -1, -3, -5