April 5

Monday, April 5, 2021 2:22 PM

Comp Sci 212

1. Induction
2. Pitfalls of Induction
3. Strong Induction

Induction

- Prove $P(0)$ - Base case
- Prove that $P(n)$ implies $P(n+1)$ for all non-negative integers n. - Inductive step
- Then $P(n)$ is true for all n

Is $P(5)$ true?

$P(0) \to P(1) \to P(2) \to P(3) \to P(4) \to P(5)$

base case

inductive step

Thm. If $P(0)$ is true and $P(n) \to P(n+1)$ for all non-negative integers n, then $P(n)$ is true for all non-negative integers n.

(Induction works).

Proof. Use contradiction.

Let n be the smallest integer for which $P(n)$ is false. (axiom).

$n \neq 0$, because $P(0)$ is true.

Know that $P(n-1)$ is true, but $P(n-1)$ implies $P(n)$, a contradiction. □

Thm. Let $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ (Fibonacci numbers).

Then

$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$

$F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$ $F_4 = 3$ $F_5 = 5$ $F_6 = 8$

$F_0 + F_1 + F_2 + F_3 + F_4 = 7$, $= F_6 - 1 = 8 - 1$

Proof. Use induction

Base case: $n=0$, $F_0 = 0 = F_2 - 1 = 1 - 1$

Inductive step: Assume $F_0 + \cdots + F_n = F_{n+2} - 1$

$\underbrace{F_0 + F_1 + \cdots + F_n}_{\text{by inductive hypothesis}} + F_{n+1} = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$ □

Thm. If $n \geq 4$, $2^n \leq 1 \cdot 2 \cdot 3 \cdots n$ ($n!$) factorial

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120
6	64	720

Proof. Use induction.

Base case: $n=4$, $2^4 = 16 \leq 4! = 24$ ✓

Inductive step: Assume $2^n \leq n!$

$2^{n+1} = 2^n \cdot 2 \leq n! \cdot 2 \leq n! \cdot (n+1) = (n+1)!$

(Inductive hypothesis) $n \geq 4$. □

Pitfalls of Induction

Thm. Any group of $n \geq 1$ people have the same name.

Proof. Use Induction

Base case: $n=1$, obvious

Inductive. Assume statement is true for n.

Number everyone from 1 to $n+1$.

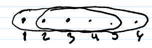
1 2 3 4 ... $n+1$.

same name

People from 1 to n all have same name.

People from 2 to $n+1$ all have same name

Both by inductive hypothesis. Together, this implies that everyone from 1 to $n+1$ have the same name □



$P(2) \to P(3)$

$P(3) \to P(4)$

$P(1)$ is true

$P(n)$ implies $P(n+1)$ if $n \geq 2$.

Issue $P(1)$ does not imply $P(2)$.

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Strong Induction

- Prove $P(0)$ - Base case
- Prove that $P(0), P(1), P(2), \ldots, P(n)$ together imply $P(n+1)$.

Theorem. Every integer $n \geq 2$ can be written as a product of primes.

Ex. $n=5$ $n=15=5 \cdot 3$

$n=7$ $n=24=3 \cdot 2 \cdot 2 \cdot 2$

prime → Use Induction composite

Proof. Base case: $n=2$, obvious

Inductive step: $P(2), \ldots, P(n-1)$ imply $P(n)$

Assume $2, \ldots, n-1$ can all be written as a product of primes.

If n is prime - use n

If n is not prime, divisible by $a \neq n$ or 1.

$n = a \cdot \left(\frac{n}{a}\right)$, since $a, \frac{n}{a} < n$, can be written as a product of primes by inductive hypothesis, therefore so can n. □