April 2

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Comp Sci 212

1. Iff
2. Contradiction
3. Induction

Iff → Proving both implication and converse

If P then Q   If Q then P

P if and only if Q → both If P then Q and If Q then P

P iff Q

Theorem. For $a, b \geq 0$,

$\frac{a+b}{2} = \sqrt{ab}$ iff $a = b$.

If $a=b$ then $\frac{a+b}{2} = \sqrt{ab}$

Proof. If $a=b$, then $\frac{a+b}{2} = a$, $\sqrt{ab} = a$ so both are equal.

If $\frac{a+b}{2} = \sqrt{ab}$ then $a=b$

If $\frac{a+b}{2} = \sqrt{ab} \xleftrightarrow{\text{then}} a^2+2ab+b^2 = 4ab \longleftrightarrow$

$a^2-2ab+b^2=0 \longleftrightarrow (a-b)^2 = 0 \longleftrightarrow$

$a-b=0 \longleftrightarrow a=b$.

(after $\sqrt{\ }$)

□

Can combine iff's

(in above proof, only need the second part.)

Def. x is even if $\frac{x}{2}$ is an integer

Def. x is odd if $\frac{x-1}{2}$ is an integer.

Thm. Let x be an integer. Then x is even iff x^2 is even.

Proof. If x is even, $\frac{x}{2} = k$ where k is an integer, therefore $\frac{x^2}{4} = k^2$, or $\frac{x^2}{2} = 2k^2$.

Therefore, x^2 is even, since $2k^2 = \frac{x^2}{2}$ is an integer.

For other direction, do contrapositive, if x is odd, then x^2 is odd.

If x^2 is even then x is even.

If x is not even then x^2 is not even

If x is odd then x^2 is odd

If x is odd, $\frac{x-1}{2} = k$ where k is an integer,

therefore $\frac{x^2-2x+1}{4} = k^2$ i.e. $\frac{x^2-1}{2} = 2k^2+x-1$.

$\frac{x^2-2x+1}{4} = k^2$

$\frac{x^2-2x+1}{2} = 2k^2$

$+x-1$ $+x-1$

$2k^2+x-1$ is an integer, so x^2 is odd.

Contradiction. P is true → If not P then Q

If not P then not Q

Theorem. $\sqrt{2}$ is irrational.

Proof. Use contradiction, assume $\sqrt{2}$ is rational, $\sqrt{2} = \frac{m}{n}$ for some integers m & n.

↳ not P

Can assume m & n are in lowest terms, and are not both even.

If not P then Q

Also have $2n^2 = m^2$. Because n^2 is an integer, m^2 is even.

Because m^2 is even so is m.

$\frac{m}{2} = k$ for some integer k, so $\frac{m^2}{4} = k^2$ for some integer k. So $\frac{2n^2}{4} = k^2$, $\frac{n^2}{2} = k^2$.

Therefore n^2 is even, and n is even.

If not P then not Q

Implication If P then Q

Contrapositive If not Q then not P

Induction

Theorem. For all non-negative integers n,

$1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$n=1$ 1 $\frac{1\cdot 2}{2} = 1$

$n=2$ $1+2=3$ $\frac{2\cdot 3}{2} = 3$

$n=3$ $1+2+3=6$ $\frac{3\cdot 4}{2} = 6$

$n=4$ $1+2+3+4=10$ $\frac{4\cdot 5}{2} = 10$

$n=5$ $1+2+3+4+5=15$ $\frac{5\cdot 6}{2} = 15$

$n=6$ $\underbrace{1+2+3+4+5}+6$

$= \frac{5\cdot 6}{2} + 6 = \frac{5\cdot 6 + 2\cdot 6}{2} = \frac{7\cdot 6}{2}$

Idea: Use solution for smaller n to prove solution for larger n.

Strategy: Have predicate $P(n)$ n is non-negative integer.

Prove $P(0)$ → Base case

Prove for all n, if $P(n)$ then $P(n+1)$ – Inductive step

Then $P(n)$ is true for all n.

Justify on Monday

Proof. Use Induction.

Base case $n=0$ $0 = \frac{0\cdot 1}{2}$ ($1 = \frac{1\cdot 2}{2}$ also works).

Inductive step If $1+2+3+4\cdots n = \frac{n(n+1)}{2}$, then

$1+2+3+4+\cdots+(n+1) = \frac{(n+1)(n+2)}{2}$.

$1+2+\cdots+n+(n+1) = \frac{n(n+1)}{2} + (n+1)$

$= \frac{n(n+1) + 2\cdot(n+1)}{2}$

$= \frac{(n+1)(n+2)}{2}$

□