Instructor: Shravas Rao

Week 9 Questions

Student: FIRSTNAME LASTNAME

Lecture 16

1. (O 7.24) This problem is similar to Exercise 7.23 in that it shows you may assume that Dictator-vs.-No-Notables tests are testing "smoothed" functions of the form $T_{1-\delta}h$ for $h : \{-1,1\}^n \to [-1,1]$, so long as you are willing to lose $O(\delta)$ in the probability that dictators are accepted.

   (a) Let U be an (α, β)-Dictator-vs.-No-Notables test using an arity-r predicate set Ψ (over domain $\{-1,1\}$) which works under the assumption that the function $f : \{-1,1\}^n \to [-1,1]$ being tested is of the form $T_{1-\delta}h$ for $h : \{-1,1\}^n \to [-1,1]$. Modify U as follows: whenever it is about to query $f(x)$, let it draw $y \sim N_{1-\delta}(x)$ and use $f(y)$ instead. Call the modified test U'. Show that the probability U' accepts an arbitrary $h : \{-1,1\}^n \to [-1,1]$ is equal to the probability U accepts $T_{1-\delta}h$.

   Solution:

   (b) Prove that U' is an $(\alpha, \beta - r\delta/2)$-Dictator-vs.-No-Notables test using predicate set Ψ.

   Solution:

2. (O 7.26) Show that when using Theorem 7.40, it suffices to have a "Dictators-vs.-No-Influentials test", meaning replacing $\mathrm{Inf}_i^{(1-\varepsilon)}[f]$ in Definition 7.37 with just $\mathrm{Inf}_i[f]$. (Hint: Exercise 7.24.)

   Solution:

3. (O 7.28) In this problem you will show that Corollary 7.43 actually follows directly from Corollary 7.44.

   (a) Consider the $\mathbb{F}_2$-linear equation $v_1 + v_2 + v_3 = 0$. Exhibit a list of 4 clauses (i.e., logical ORs of literals) over the variables such that if the equation is satisfied, then so are all 4 clauses, but if the equation is not satisfied, then at most 3 of the clauses are. Do the same for the equation $v_1 + v_2 + v_3 = 1$

   Solution:

   (b) Suppose that for every $\delta > 0$ there is an efficient algorithm for $(\frac{7}{8} + \delta, 1 - \delta)$-approximating Max-E3-Sat. Give, for every $\delta > 0$, an efficient algorithm for $(\frac{1}{2} + \delta, 1 - \delta)$-approximating Max-E3-Lin.

   Solution:

   (c) Alternatively, show how to transform any (α, β)-Dictator-vs.-No-Notables test using Max-E3-Lin predicates into a $(\frac{3}{4} + \frac{1}{4}\alpha, \beta)$-Dictator-vs.-No-Notables test using Max-E3-Sat predicates.

Solution:

Lecture 18

1. (O 9.8) Fix $k \in N$. The goal of this exercise is to show that "projection to degree k is a bounded operator in all L_p norms, $p > 1$". Let $f : \{-1,1\}^n \to \mathbb{R}$.

   (a) Let $q \geq 2$. Show that $\|f^{\leq k}\|_q \leq \sqrt{q-1}^k \|f\|_q$. (Hint: Use Theorem 9.21 to show the stronger statement $\|f^{\leq k}\|_q \leq \sqrt{q-1}^k \|f\|_2$.)

   Solution:

   (b) Let $1 < q \leq 2$. Show that $\|f^{\leq k}\|_q \leq (1/\sqrt{q-1})^k \|f\|_q$. (Hint: Either give a similar direct proof using the $(p,2)$-Hypercontractivity Theorem, or explain how this follows from part (a) using the dual norm Proposition 9.19.)

   Solution:

2. (O 9.11)

   (a) Suppose $\mathbb{E}[X] = 0$. Show that X is $(q,q,0)$-hypercontractive for all $q \geq 1$. (Hint: Use monotonicity of norms to reduce to the case $q = 1$.)

   Solution:

   (b) Show further that X is (q,q,ρ)-hypercontractive for all $0 \leq \rho < 1$. (Hint: Write $(a+\rho X) = (1-\rho)a + \rho(a+X)$ and employ the triangle inequality for $\|\cdot\|_q$.)

   Solution:

3. (O 9.17) Deduce the $p \leq 2 \leq q$ cases of the Hypercontractivity Theorem from the $(2,q)$- and $(p,2)$Hypercontractivity Theorems. (Hint: Use the semigroup property of T_ρ, Exercise 2.32.)

Solution: