Lecture 14

1. (O 7.11)

   (a) Let ϕ be a CNF of size s and width $w \geq 3$ over variables $x_1, \dots, x_n$. Show that there is an "equivalent" CNF ϕ' of size at most $(w-2)s$ and width 3 over the variables $x_1, \dots, x_n$ plus auxiliary variables $\Pi_1, \dots, \pi_\ell$ with $\ell \leq (w-3)s$. Here "equivalent" means that for every x such that $\phi(x) = \mathsf{True}$ there exists Π such that $\phi'(x, \Pi) = \mathsf{True}$; and, for every x such that $\phi(x) = \mathsf{False}$ we have $\phi'(x, \Pi) = \mathsf{False}$ for all Π.

   Solution:

   (b) Extend the above so that every clause in ϕ' has width exactly 3 (the size may increase by $O(s)$).

   Solution:

2. (O 7.12) Suppose there exists an r-query PCPP reduction $\mathcal{R}_1$ with rejection rate λ. Show that there exists a 3-query PCPP reduction $\mathcal{R}_2$ with rejection rate at least $\lambda/(r2^r)$. The proof length of $\mathcal{R}_2$ should be at most $r2^r \cdot m$ plus the proof length of $\mathcal{R}_1$ (where m is the description-size of $\mathcal{R}_1$'s output) and the predicates output by the reduction should all be logical ORs applied to exactly three literals.

   Solution:

3. (O 7.17) Give a 3-query, length-$O(n)$ PCPP system (with rejection rate a positive universal constant) for the class $\{w_1, w_2 \in \mathbb{F}_2^n : (-1)^{w_1 \cdot w_2} = 1\}$.

   Solution:

Lecture 15

1. (O 7.10)

   (a) Say that A is an (α, β)-distinguishing algorithm for Max-CSP(Ψ) if it outputs 'YES' on instances with value at least β and outputs 'NO' on instances with value strictly less than α. (On each instance with value in $[\alpha, \beta)$, algorithm A may have either output.) Show that if there is an efficient (α, β)-approximation algorithm for Max-CSP(Ψ), then there is also an efficient (α, β)-distinguishing algorithm for Max-CSP(Ψ).

   Solution:

(b) Consider Max-CSP(Ψ), where Ψ be a class of predicates that is closed under restrictions (to nonconstant functions); e.g., Max-3-Sat. Show that if there is an efficient $(1, 1)$-distinguishing algorithm, then there is also an efficient $(1, 1)$-approximation algorithm. (Hint: Try out all labels for the first variable and use the distinguisher.)

Solution:

2. (O 7.20) A randomized assignment for an instance $\mathcal{P}$ of a CSP over domain is a mapping F that labels each variable in V with a probability distribution over domain elements. Given a constraint (S, ψ) with $S = (v_1, \ldots, v_r)$, we write $\psi(F(S)) \in [0, 1]$ for the expected value of $\psi(F(v_1), \ldots, F(v_r))$. This is simply the probability that ψ is satisfied when one actually draws from the domain-distributions assigned by$F$. Finally, we define the value of F to be $\mathrm{Val}_{\mathcal{P}}(F) = E_{S,\psi}[\psi(F(S))]$.

(a) Suppose that A is a deterministic algorithm that produces a randomized assignment of value α on a given instance $\mathcal{P}$. Show a simple modification to A that makes it a randomized algorithm that produces a (normal) assignment whose value is α in expectation. (Thus, in constructing approximation algorithms we may allow ourselves to output randomized assignments.)

Solution:

(b) Let A be the deterministic Max-E3-Sat algorithm that on every instance outputs the randomized assignment that assigns the uniform distribution on $\{0, 1\}$ to each variable. Show that this is a $(\frac{7}{8}, \beta)$-approximation algorithm for any β. Show also that the same algorithm is a $(\frac{1}{2}, \beta)$-approximation algorithm for Max-3-Lin.

Solution:

(c) When the domain Ω is $\{-1, 1\}$, we may model a randomized assignment as a function $f : V \to [-1, 1]$; here $f(v) = \mu$ is interpreted as the unique probability distribution on $\{-1, 1\}$ which has mean μ. Now given a constraint (S, ψ) with $S = (v_1, \ldots, v_r)$, show that the value of f on this constraint is in fact $\psi(f(v_1), \ldots, f(v_r))$, where we identify $\psi : \{-1, 1\}^r \to \{0, 1\}$ with its multilinear (Fourier) expansion. (Hint: Exercise 1.4.)

Solution:

(d) Let Ψ be a collection of predicates over domain $\{-1, 1\}$. Let $\nu = \min_{\psi \in \Psi}\{\widehat{\psi}(\emptyset)\}$. Show that outputting the randomized assignment $f = 0$ is an efficient $\nu, beta)$-approximation algorithm for Max-CSP(Ψ).

Solution:

3. (O 7.21) Let F be a randomized assignment of value α for CSP instance $\mathcal{P}$ (as in Exercise 7.20). Give an efficient deterministic algorithm that outputs a usual assignment of value at least α. (Hint: Try all possible labelings for the first variable and compute the expected value

that would be achieved if F were used for the remaining variables. Pick the best label for the first variable and repeat.)

Solution: