February 23

Tuesday, February 23, 2021 1:33 PM

Boolean Functions

1. Summary
   - Bonami Lemma
   - Hypercontractivity
   - KKL
2. Q+A
3. Problems

Announcements
- Week 6 problems due March 2
- 7.1, 9.1 for Thursday

Bonami Lemma

$f: \{-1,1\}^n \to \mathbb{R}$, degree-$k$

$\|f\|_4^4 \le 9^k \cdot \|f\|_2^4$

Corollary $\|T_{\frac{1}{\sqrt{3}}} f^{=k}\|_4 \le \|f^{=k}\|_2$

$(2,4)$ Hypercontractivity Theorem

$\|T_{\frac{1}{\sqrt{3}}} f\|_4 \le \|f\|_2$

Proof.

$T_{\frac{1}{\sqrt{3}}} f = x_n \frac{1}{\sqrt{3}} D_n Tf + E_n \cdot Tf,$

Similar to Bonami

$(2,q)$ can be generalized

Fact. $\|T_{\frac{1}{\sqrt{3}}} \cdot f\|_2 = \mathrm{Stab}_{\frac{1}{3}}[f]^{\frac{1}{2}}$

uses Hölder $\le \|f\|_{4/3}$

$(\frac{4}{3}, 2)$ Hypercontractivity

Hölder - $\langle f, g\rangle \le \|f\|_p \cdot \|g\|_q$ if $\frac{1}{p} + \frac{1}{q} = 1$

$p=2, q=2$, Cauchy-Schwarz

$p=1, q=\infty$, $\langle f, g\rangle \le \|f\|_1 \cdot \max|g_i|$

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old inner product → new inner product: divide by n

old p-norm → new p-norm: divide by $n^{\frac{1}{p}}$

Corollary Let $f: \{-1,1\}^n \to \{0,1\}$

$\mathbb{E}[f] = \alpha$

$\mathrm{Stab}_{\frac{1}{3}}[f] \le \alpha^{\frac{3}{2}}$ $\quad \|f\|_{\frac{4}{3}} = \alpha^{\frac{3}{4}}$

still works if → $\{-1,0,1\}$

Corollary. Let $f: \{-1,1\}^n \to \{-1,1\}$

$\alpha = \mathrm{Inf}_i[f]$

$\mathrm{Stab}_{\frac{1}{3}}[D_i f] \le \alpha^{\frac{3}{2}}$

KKL - $\mathrm{MaxInf}[f] \ge \Omega(\mathrm{Var}[f]) \frac{\log n}{n}$

$I[f] \ge \Omega(\mathrm{Var}[f]) \log_{\frac{q}{\mathrm{MaxInf}[f]}}$

Friedgut's Thm - $\Lambda = \frac{2I[f]}{\varepsilon}$,

$J = \{j : \mathrm{Inf}_j[f] \ge \exp(-3\Lambda)\}$,

f is ε-concentrated on $\mathcal{J} = \{S \subseteq J, |S| \le \Lambda\}$.

KKL, Tribes

Problem 1 - Bonami, $k=1$

Problem 2 - necessity 9^k combinatorics

Problem 3 - ± 1 not r.v.s, B-reasonable

Problem 4 - why is $\{-1,1\}$ necessary for KKL

$[-1,1]$

Breakout room : - Problem 1