Lecture 12

1. (O 9.2) For $k = 1$, improve the 9 in the Bonami Lemma to 3. More precisely, suppose $f : \{-1,1\}^n \to \mathbb{R}$ has degree at most 1 and that $x_1, \ldots, x_n$ are independent 3-reasonable random variables satisfying $\mathbb{E}[x_1] = \mathbb{E}[x_i^3] = 0$. (For example,the x_i's may be uniform ± 1 bits.) Show that $f(x)$ is also 3-reasonable. (Hint: By direct computation, or by running through the Bonami Lemma proof with $k = 1$ more carefully.)

   Solution:

2. (O 9.3) Let k be a positive multiple of 3 and let $n \geq 2k$ be an integer. Define $f : \{-1,1\}^n \to \mathbb{R}$ by
$$f(x) = \sum_{\substack{S \subseteq [n] \\ |S| = k}} x^S.$$

   (a) Show that
$$\mathbb{E}[f^4] \geq \frac{\binom{n}{k/3,k/3,k/3,k/3,k/3,k/3,n-2k}}{\binom{n}{k}^2} \mathbb{E}[f^2]^2$$
   where the numerator of the fraction is a multinomial coefficient – specifically, the number of ways of choosing six disjoint size-$k/3$ subsets of $[n]$. (Hint: Given such size-$k/3$ subsets, consider quadruples of size-k subsets that hit each size-$k/3$ subset twice.)

   Solution:

   (b) Using Stirling's Formula, show that
$$\lim_{n \to \infty} \frac{\binom{n}{k/3,k/3,k/3,k/3,k/3,k/3,n-2k}}{\binom{n}{k}^2} = \Theta(k^{-2} 9^k).$$
   Deduce the following lower bound for the Bonami Lemma: $\|f\|_4 \geq \Omega(k^{-1/2}) \cdot \sqrt{3}^k \|f\|_2$.

   Solution:

3. (O 9.4) Let $x_1, \ldots, x_n$ be independent, not necessarily identically distributed, random variables satisfying $\mathbb{E}[x_1] = \mathbb{E}[x_i^3] = 0$. (This holds if, e.g., each $-x_i$ has the same distribution as x_i.) Assume also that each x_i is B-reasonable. Let $f = F(x_1, \ldots, x_n)$, where F is a multilinear polynomial of degree at most k. Then f is $\max\{B, 9\}^k$-reasonable.

   Solution:

4. (O 9.20) Show that the KKL Theorem fails for functions$f : \{-1,1\}^n \to [-1,1]$, even under the assumption $\mathrm{Var}[f] \geq \Omega(1)$. (Hint: $f(x) = \mathrm{trunc}_{[-1,1]}(\frac{x_1 + \cdots + x_n}{\sqrt{n}})$.)

Solution:

Lecture 13

1. (O 7.1) Suppose there is an r-query local tester for property $\mathcal{C}$ with rejection rate λ. Show that there is a testing algorithm that, given inputs $0 < \varepsilon, \delta \leq 1/2$ makes $O(\frac{r \log(1/\delta)}{\lambda\varepsilon})$ (nonadaptive) queries to f and satisfies the following:
   - If $f \in \mathcal{C}$, then the tester accepts with probability 1.
   - If f is ε-far from $\mathcal{C}$, then the tester accepts with probability at most δ.

   Solution:

2. (O 7.5) Let $\mathcal{O} = \{w \in \mathbb{F}_2^n : w \text{ has an odd number of 1's}\}$. Let T be any $n-1$-query string testing algorithm that accepts every $w \in \mathcal{O}$ with probability 1. Show that T in fact accepts every string $v \in \mathbb{F}_2^n$ with probability 1 (even though $\text{dist}(w, \mathcal{O}) = \frac{1}{n} > 0$ for half of all strings w). Thus locally testing O requires n queries.

   Solution:

3. (O 7.6) Let T be a 2-query testing algorithm for functions $\{-1,1\}^n \to \{-1,1\}$. Suppose that T accepts every dictator with probability 1. Show that it also accepts $\text{Maj}_{n'}$ with probability 1 for every odd $n' < n$. This shows that there is no 2-query local tester for dictatorship assuming $n = 2$. (Hint: You'll need to enumerate all predicates on up to 2 bits.)

   Solution: