February 18

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Boolean Functions

1. Summary
   - Level-1 & $\frac{2}{\pi}$ Theorems
   - Reasonable random variables
2. Q&A
3. Summary.

Level-1 Thm. Let $f: \{-1,1\}^n \to \{0,1\}$ with $\alpha = \mathbb{E}[f] \le \frac{1}{2}$. Then,

$W^1[f] \le O(\alpha^2 \log \frac{1}{\alpha})$. — α is small

$\frac{2}{\pi}$ Theorem. $f: \{-1,1\}^n \to \{-1,1\}$. — $\alpha \approx \frac{1}{2}$, $\mathbb{E}[f] \approx 0$.

$|\hat{f}(i)| \le \varepsilon \ \forall i \in [n]$. Then,

$W^1[f] \le \frac{2}{\pi} + O(\varepsilon)$.

If $W^1[f] \ge \frac{2}{\pi} - \varepsilon$, then f is $O(\sqrt{\varepsilon})$-close to $\mathrm{sgn}(f^{=1})$.

Level-1 → Hoeffding/Chernoff → tails

$\frac{2}{\pi}$ - → Berry-Esseen → bulk

$L(x) = \sum_{i=1}^{n} \hat{f}(i) \cdot x_i$ range is $\{0,1\}$

$W^1[f] = \mathbb{E}[f(x)L(x)]$

$\le \mathbb{E}[\mathbb{1}_{L(x)\ge t} \cdot L(x)]$

t is s.t. $\Pr[L(x) \ge t] \approx \alpha$

Level-1 Thm. $= \mathbb{E}[L(x) \mid L(x) \ge t] \cdot \Pr[L(x) \ge t]$

$= \mathbb{E}[L(x) \mid L(x) \ge t] \cdot \alpha$

If $W^1[f] \ge \frac{2}{\pi} - \varepsilon$ range $\{-1,1\}$

$\langle f, L \rangle \approx \mathbb{E}[|L|]$

$n = 3$, size of domain is 8

$L(x)$ - 8 - 7 - 6 = 5 = 4 3 2 1

$\frac{8+7}{8} =$

$\left(\frac{8+7}{2}\right) \cdot \frac{1}{4}$

f - range $\{0,1\}$

$\alpha = \frac{1}{4}$, two x's s.t. $f(x) = 1$

Def - Random variable is β-reasonable if $\mathbb{E}[X^4] \le \beta \mathbb{E}[X^2]^2$

$\|f\|_4$ vs. $\|f\|_2$

Concentration - $\Pr[X \ge t\|X\|_2] \le \frac{\beta}{t^4}$

(Markov's inequality)

Anti-concentration $\Pr[X \ge t\|X\|_2] \ge \frac{(1-t^2)^2}{\beta}$

(2nd moment method)

2nd moment method → If $X \ge 0$

$\Pr[X > 0] \ge \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}$

X is non-zero

Jensen $\mathbb{E}[X]^2 \le \mathbb{E}[X^2]$

Proof. $\mathbb{E}[X] = \mathbb{E}[X \cdot \mathbb{1}_{\{X>0\}}]$

$\le \mathbb{E}[X^2]^{1/2} \cdot \mathbb{E}[\mathbb{1}_{\{X>0\}}^2]^{1/2}$

(Cauchy-Schwarz)

$\Pr[X > 0]$ □

Union bound - $\Pr[E_1 \text{ or } E_2 \text{ or } \cdots \text{ or } E_n] \le \sum_{i=1}^{n} \Pr[E_n]$

$R = \sum \mathbb{1}_{E_i}$ $\Pr[R > 0]$

r.v. 1 if E_i, 0 ow.

(PIE is another lower-bound)

Concentration $\Pr[X \ge t] = \Pr[X^2 \ge t^2]$

$\le \frac{\mathbb{E}[X^2]}{t^2}$

derive Chernoff, Hoeffding

$\frac{1}{q}$ $\frac{q-1}{q}$

$\mathbb{E}[X] \le \mathbb{E}[X^q]^{1/q} \cdot \mathbb{E}[\mathbb{1}_{\{X>0\}}^{\frac{q}{q-1}}]^{\frac{q-1}{q}}$

(Hölder)

$\Pr(X > 0]^{q-1} \ge \frac{\mathbb{E}[X]^q}{\mathbb{E}[X^q]}$

$\Pr[X > 0] \ge \frac{\mathbb{E}[X]^{\frac{q}{q-1}}}{\mathbb{E}[X^q]^{\frac{1}{q-1}}}$ → not sure if this works

Problem 1 - I wrote no guarantees that it's right - generalize Room 1, 4

Problem 2 - Reasonable after shifting Level-1, $\frac{2}{\pi}$ Room 2, 5

Problem 3 - Construction → X w/ $1+X$ Room 3, 6

Odd's $\mathbb{E}[X^2]$ much larger than $\mathbb{E}[X]$