

Boolean Functions

1. Summary

- Level-1 & $\frac{2}{\pi}$ Theorems
- Reasonable random variables

2. Q+A

3. Summary.

Level-1 Thm. Let $f: \{-1, 1\}^n \rightarrow \{0, 1\}$
with $d = \mathbb{E}[f] \leq \frac{1}{2}$. Then,

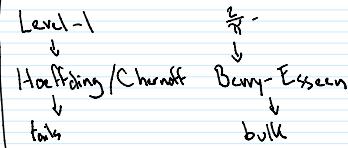
$$W[f] \leq O(d^2 \log \frac{1}{2}). \quad - d \text{ is small}$$

$\frac{2}{\pi}$ Theorem. $f: \{-1, 1\}^n \rightarrow \{0, 1\}$. $- d \approx \frac{1}{2}$
 $\mathbb{E}[f] \approx 0$.

$$|\hat{f}(i)| \leq \epsilon \quad \forall i \in [n]. \text{ Then,}$$

$$W[f] \leq \frac{2}{\pi} + O(\epsilon).$$

If $W[f] \geq \frac{2}{\pi} - \epsilon$, then f is $O(\epsilon)$ -close to $\text{sgn}(f^\wedge)$.



$$L(x) = \sum_{i=1}^n \hat{f}(i)x_i \quad \text{range is } \{0, 1\}$$

$$W[f] = \mathbb{E}[f(x)L(x)]$$

$$\leq \mathbb{E}[\mathbf{1}_{L(x) \geq f} \cdot L(x)]$$

$$+ \text{ is s.t. } \Pr[L(x) \geq f] \approx d$$

$$\text{Level-1} = \mathbb{E}[\mathbf{1}_{L(x)} | L(x) \geq f] \cdot \Pr[L(x) \geq f]$$

$$\text{Thm.} = \mathbb{E}[\mathbf{1}_{L(x)} | L(x) \geq f] \cdot 2$$

$$\text{If } W[f] \geq \frac{2}{\pi} - \epsilon \quad \text{range } \{0, 1\}$$

$$\langle f, L \rangle \approx \mathbb{E}[|L|]$$

$$\begin{array}{l} n=3, \text{ size of domain is } 3 \\ \text{f - range } \{0, 1\} \\ L(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ 2 & x = 2 \\ 3 & x = 3 \end{cases} \\ \mathbb{E}[L] = \frac{3}{3} = 1 \\ \mathbb{E}[L^2] = \frac{3+2+1}{3} = \frac{6}{3} = 2 \\ \mathbb{E}[L^3] = \frac{3+4+3+1}{3} = \frac{11}{3} \end{array}$$

Def. Random variable is β -reasonable
if $\mathbb{E}[X^4] \leq \beta \mathbb{E}[X^2]^2$

$\|f\|_1$ vs. $\|f\|_2$

$$\text{Concentration: } \Pr[X \geq t \|X\|_2] \leq \frac{\beta}{t^4}$$

(Markov's inequality)

$$\text{Anti-concentration: } \Pr[X \geq t \|X\|_2] \geq \frac{(1-t)^2}{\beta}$$

(2nd moment method)

2nd moment method \rightarrow If $X \geq 0$

$$\Pr[X > 0] \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}$$

Jensen $\mathbb{E}[X^2] \leq \mathbb{E}[X^2]$

$$\text{Prof. } \mathbb{E}[X] = \mathbb{E}[X \cdot \mathbf{1}_{\{X > 0\}}]$$

$$\leq \mathbb{E}[X]^2 \cdot \mathbb{E}[\mathbf{1}_{\{X > 0\}}]^{1/2}$$

(Cauchy-Schwarz)

$$\Pr[X > 0]$$

$$\text{Union bound: } \Pr[E_1 \cup E_2 \cup \dots \cup E_n] \leq \sum_{i=1}^n \Pr[E_i]$$

$$R = \sum \mathbf{1}_{E_i} \quad \Pr[R > 0]$$

$\Pr[R > 0]$
0 or 1 if E_i

(PIE is another lower-bound)

$$\text{Concentration: } \Pr[X \geq f] = \Pr[X^2 \geq f^2]$$

$$\leq \frac{\mathbb{E}[X^2]}{f^2}$$

$$\frac{1}{f^2} \geq \frac{\mathbb{E}[X^2]}{f^2} \quad \text{drive Chernoff, Hoeffding}$$

$$\mathbb{E}[X] = \mathbb{E}[X^2]^{1/2} \cdot \mathbb{E}[\mathbf{1}_{\{X > 0\}}]^{1/2}$$

(Holder)

$$\Pr[X > 0]^{1/2} \geq \frac{\mathbb{E}[X]}{\mathbb{E}[X^2]}$$

$$\Pr[X > 0] \geq \frac{\mathbb{E}[X]^{1/2}}{\mathbb{E}[X^2]^{1/2}}$$

not sure if this works

Problem 1 - I wrote no guarantees that its right - generalize

Problem 2 - Reasonable after shifting

Problem 3 - Construction

↓ Odds $\mathbb{E}[X^2]$ much larger
than $\mathbb{E}[X]$

Room 1, 4

Room 2, 5

Room 3, 4