Lecture 10

1. (O 2.43) Show that for all functions $f : \{-1,1\}^n \to \{-1,1\}$

$$\frac{1-e^{-2}}{2n}\mathbf{I}[f] \leq NS_{1/n}[f] \leq \frac{1}{n}\mathbf{I}[f].$$

Solution:

2. (O 5.30) Consider the sequence of LTFs $f_n : \{-1,1\}^n \to \{0,1\}$ defined by $f_n(x) = 1$ if and only if $\sum_{i=1}^n \frac{1}{\sqrt{n}} x_i > t$. (I.e., f_n is the indicator of the Hamming ball of radius $\frac{n}{2} - \frac{t}{2}\sqrt{n}$ centered at $(1, 1, \ldots, 1)$.) Show that

$$\lim_{n\to\infty} \mathbb{E}[f_n] = \overline{\Phi}(t), \qquad \lim_{n\to\infty} \mathbf{W}^1[f_n] = \phi(t)^2,$$

where ϕ is the pdf of a standard Gaussian and $\overline{\Phi}$ is the complementary cdf (i.e., $\overline{\Phi}(u) = \int_u^\infty \phi$). (Hint for $\mathbf{W}^1[f_n]$: use symmetry to show it equals the square of $\mathbb{E}[f_n(x)\sum \frac{1}{\sqrt{n}} x_i]$.)

Solution:

3. (O 5.32) Consider the sequence of LTFs defined in the previous problem. Show that

$$\lim_{n\to\infty} Stab_\rho[f_n] = \Pr[z_1 > t, z_2 > t]$$

where z_1 and z_2 are standard Gaussians with correlation $\mathbb{E}[z_1 z_2] = \rho$.

Solution:

4. (O 5.28) For integer $0 \leq j \leq n$, define $\mathscr{K}_j : \{-1,1\}^n \to \mathbb{R}$ by $\mathscr{K}_j(x) = \sum_{|S|=j} x^S$. Since $\mathscr{K}_j$ is symmetric, the value $\mathscr{K}_j(x)$ depends only on the number z of -1's in x; or equivalently, on $\sum_{i=1}^n x_i$. Thus we may define $K_j : \{0, 1, \ldots, n\} \to \mathbb{R}$ by $K_j(z) = \mathscr{K}_k(x)$ for any x with $\sum_i x_i = n - 2z$.

   (a) Show that $K_j(z)$ can be expressed as a degree-j polynomial in z. It is called the *Kravchuk* (or *Krawtchouk*) *polynomial* of degree j. (The dependence on n is usually implicit.)

   Solution:

   (b) Show that

$$\sum_{j=0}^n \mathscr{K}_j(x) = \begin{cases} 2^n & \text{if } x = (1,1,\ldots,1) \\ 0 & \text{otherwise} \end{cases}$$

Solution:

(c) Show for $\rho \in [-1, 1]$ that $\sum_{j=0}^{n} \mathscr{K}_j(x)\rho^j = 2^n \Pr[y = (1, 1, \dots, 1)]$, where $y \sim N_p(x)$.

Solution:

(d) Deduce the generating function identity, that $K_j(z)$ is the coefficient of ρ^j in $(1-\rho)^x(1+\rho)^{n-x}$.

Solution:

Lecture 11

1. Let $S_1, \dots, S_t$ be a collection of disjoint subsets of $[n]$ such that $|\widehat{f}(S_i)| \le \varepsilon$ for all i. Give appropriate generalizations of the level-1 inequality and $2/\pi$ theorem.

Solution:

2. (O 9.1) For every $1 < b < B$ show that there is a b-reasonable random variable X such that $1 + X$ is not B-reasonable.

Solution:

3. Show that if X is a nonnegative random variable with $\Pr[X > K] = \delta$ and $\mathbb{E}[X] \ge L > K$ then $\mathbb{E}[X^2] \ge (L - K)^2/\delta$.

Solution: