Lecture 8

1. (O 4.17) Let $f : \{\mathsf{True}, \mathsf{False}\}^n \to \{\mathsf{True}, \mathsf{False}\}$ be computable by a CNF C of width w. In this exercise you will show that $\mathbf{I}[f] \le w$. Consider the following randomized algorithm that tries to produce an input $\mathbf{x} \in f^{-1}(\mathsf{True})$. First, choose a random permutation $\pi \in S_n$. Then for $1 = 1, \ldots, n$: If the single-literal clause $\mathbf{x}_{\pi(i)}$ appears in C, then set $\mathbf{x}_{\pi(i)} = \mathsf{True}$, syntactically simplify C under this setting, and say that coordinate $\pi(i)$ is "forced". Similarly, if the single-literal clause $\overline{\mathbf{x}}_{\pi(i)}$ appears in C, then set $\mathbf{x}_{\pi(i)} = \mathsf{False}$, syntactically simplify C, and say that $\pi(i)$ is "forced". If neither holds, set $\mathbf{x}_{\pi(i)}$ uniformly at random. If C ever contains two single-literal clauses x_j and $\overline{x}_j$, the algorithm "gives up" and outputs $\mathbf{x} = \perp$

   (a) Show that if $\mathbf{x} \ne \perp$, then $f(\mathbf{x}) = \mathsf{True}$.

   Solution:

   (b) For $x \in f^{-1}(\mathsf{True})$ let $p(x) = \Pr[\mathbf{x} = x]$. For $j \in [n]$ let I_j be the indicator random variable for the event that coordinate $j \in [n]$ is forced. Show that $p(x) = \mathbb{E}\left[\prod_{j=1}^{n} (1/2)^{1-I_j}\right]$

   Solution:

   (c) Deduce $2^n p(x) \ge 2 \sum_{j=1}^{n} \mathbb{E}[I_j]$

   Solution:

   (d) Show that for every x with $f(x) = \mathsf{True}, f(x^{\oplus j}) = \mathsf{False}$ it holds that $\mathbb{E}[I_j \mid \mathbf{x} = x] \ge 1/w$

   Solution:

   (e) Deduce $\mathbf{I}[f] \le w$.

   Solution:

2. (O 4.19)In this exercise you will prove the Baby Switching Lemma with constant 3 in place of 5. Let $\phi = T_1 \vee T_2 \vee \cdots \vee T_s$ be a DNF of width $w \ge 1$ over variables $x_1, \ldots, x_n$. We may assume $\delta \le 1/3$, else the theorem is trivial.

   (a) (a) Suppose $R = (J|z)$ is a "bad" restriction, meaning that $\phi_{J|z}$ is not a constant function. Let i be minimal such that $(T_i)_{J|z}$ is neither constantly True or False, and let j be minimal such that x_j or $\overline{x_j}$ appears in this restricted term. Show there is a unique restriction $R' = (J\backslash\{j\}|z')$ extending R that doesn't falsify T_i.

   Solution:

   (b) Suppose we enumerate all bad restrictions R, and for each we write the associated R' as in (a). Show that no restriction is written more than w times.

Solution:

(c) If $(J|z)$ is a δ-random restriction and R and R' are as in (a), show that

$$\Pr[(J|z) = R] = \frac{2\delta}{1-\delta}\Pr[(J|z) = R']$$

Solution:

(d) Complete the proof by showing $\Pr[(J|z) \text{ is bad}] \leq 3\delta w$

Solution:

Lecture 9

1. (O 5.2) Let $f(x) = \operatorname{sgn}(a_0 + a_1x_1 + \cdots + a_nx_n)$ be an LTF.

   (a) Show that if $a_0 = 0$, then $\mathbb{E}[f] = 0$. (Hint: Show that f is in fact an odd function.)

   Solution:

   (b) Show that if $a_0 \geq 0$, then $\mathbb{E}[f] \geq 0$. Show that the converse need not hold.

   Solution:

   (c) Suppose $g : \{-1,1\}^n \to \{-1,1\}$ is an LTF with $\mathbb{E}[g] = 0$. Show that g can be represented as $g(x) = \operatorname{sgn}(c_1x_1 + \cdots + c_nx_n)$

   Solution:

2. (O 5.5) Suppose $\ell : \{-1,1\}^n \to \mathbb{R}$ is defined by $\ell(x) = a_0 + a_1x_1 + \cdots + a_nx_n$. Define $\widetilde{\ell} : \{-1,1\}^{n+1} \to \mathbb{R}$ is defined by $\widetilde{\ell}(x) = a_0x_0 + a_1x_1 + \cdots + a_nx_n$. Show that $\|\widetilde{\ell}\|_1 = \|\ell\|_1$ and $\|\widetilde{\ell}\|_2^2 = \|\ell\|_2^2$.

   Solution:

3. (O 5.7) Consider the following "correlation distillation" problem (cf. Exercise 2.56). For each$i \in [n]$ there is a number $\rho_i \in [-1,1]$ and an independent sequence of pairs of ρ_i-correlated bits, $(a^{(1)}, b^{(1)}), (a^{(2)}, b^{(2)}), (a^{(3)}, b^{(3)})$, etc. Party A on Earth has access to the stream of bits $a^{(1)}, a^{(2)}, a^{(3)}, \ldots$ and a party B on Venus has access to the stream $b^{(1)}, b^{(2)}, b^{(3)}, \ldots$. Neither party knows the numbers $\rho_1, \ldots, \rho_n$. The goal is for B to estimate these correlations. To assist in this, A can send a small number of bits to B. A reasonable strategy is for A to send $f(a^{(1)}), f(a^{(2)}), f(a^{(3)})$ to B, where $f : \{-1,1\}^n \to \{-1,1\}$ is some Boolean function. Using this information B can try to estimate $\mathbb{E}[f(a)b_i]$ for each i.

(a) Show that $\mathbb{E}[f(a)b_i] = \widehat{f}(i)\rho_i$.

Solution:

(b) This motivates choosing an f for which all $\widehat{f}(i)$ are large. If we also insist all $\widehat{f}(i)$ be equal, show that majority functions f maximize this common value.

Solution:

4. (5.8) For $n \geq 2$, let $f : \{-1,1\}^n \to \{-1,1\}$ be a randomly chosen function (as in Exercise 1.7). Show that $\hat{\|}f\hat{\|}_\infty \leq 2\sqrt{n}2^{-n/2}$ except with probability at most 2^{-n}.

Hint: First, consider one Fourier coefficient. Then apply a union bound.

Solution: