January 28

Thursday, January 28, 2021 1:20 PM

Boolean Functions

1. Summary
   - Spectral Concentration
   - Decision Trees
2. Q+A
3. Problems

Given query/random access to $f: \{-1,1\}^n \to \{-1,1\}$, output $h: \{-1,1\}^n \to \{-1,1\}$ s.t. $\mathrm{dist}(h,f) \le \epsilon$.

Assume $f \in \mathcal{C}$, $\mathcal{C}$ = some class of functions.

- Ex. $\mathcal{C}$ - linear functions, n queries

Decision trees - size = # leaves

depth = longest root-to-leaf path

$\mathcal{C} = \{\text{size } s \text{ decision trees}\}$

compute h in $\mathrm{poly}(n, s, 1/\epsilon)$ queries/time

def. $\mathcal{F}$ - collection of subsets $\subseteq \mathcal{P}([n])$

f is ϵ-concentrated on $\mathcal{F}$ if $\sum_{S \notin \mathcal{F}} \hat{f}(S)^2 \le \epsilon$

$\mathcal{F} = \{S \mid |S| \le k\}$ - concentrated up to degree-k

↳ $\# = \binom{n}{k} \le n^k$

Thm. If f is ϵ-concentrated on $\mathcal{F}$, can learn f to error ϵ, in time $\mathrm{poly}(|\mathcal{F}|, n, \frac{1}{\epsilon})$

- estimate each Fourier coefficient in $\mathcal{F}$
- turn into a Boolean function

Chernoff bound

if $\mathbb{E}[X_i] = 0$, $|X_i| \le 1$

$\Pr[|X_1 + \cdots + X_n| \ge u\sqrt{n}] \le 2e^{-u^2 \cdot c}$

$u\sqrt{n} = \epsilon \cdot n$, $2e^{-u^2 c} = \delta$

$u\sqrt{n} \le \epsilon \cdot n$, $e^{-u^2 c} \le \delta$

$n \approx \left(\frac{u}{\epsilon}\right)^2$, $u \approx \sqrt{\log(\frac{1}{\delta})}$

# of samples $= n$ - $\log(\frac{1}{\delta}) \cdot \frac{1}{\epsilon^2}$

Prop. f is ϵ-concentrated up to degree $\mathbb{I}[f]/\epsilon$ ($O(\sqrt{n}/\epsilon)$ if f is monotone).

Prop. f is is ϵ-concentrated up to degree $\frac{1}{\delta}$ if $\epsilon = 3\,NS_\delta[f]$

Using ideas from Markov's inequality

Peres: $\forall$ all weighted majority, $NS_\delta[f] \le 2\sqrt{\delta}$

Question 1 → sparsity induction — 1, 5

Question 2 → sparsity $(f) \cdot$ sparsity $(\hat{f}) \ge$ — 2, 6

Question 3 → About decision trees, manipulating, — 3, 7

Question 4 → randomized rounding — 4, 8

↳ Approximation algorithms

Decision trees - decompose into sum of paths

depth k

↳ degree $\le k$

sparsity (# of non-zero Fourier coefficients) $\le \underbrace{s}_{\text{size}} \cdot 2^k \le 4^k$

$\sum_S |\hat{f}(S)| = \|\hat{f}\|_1 \le s \le 2^k$, $\hat{f}(S) = \frac{\text{integer}}{2^k}$