January 26

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Boolean Functions

1. Summary
   - Noise stability / operator
   - Arrow's theorem
2. Q&A
3. Problems

Announcements

- Week 1 solutions posted

$\mathbb{I}[f] = \sum_{S \subseteq [n]} |S| \cdot \hat{f}(S)^2$

$\mathrm{Var}[f] = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} \hat{f}(S)^2$

Poincaré Inequality - $\mathrm{Var}[f] \leq \mathbb{I}[f]$

$y \sim N_p(x)$ $\quad y_i = \begin{cases} x_i & \text{w.p. } \frac{1}{2} + \frac{1}{2}p \\ -x_i & \text{w.p. } \frac{1}{2} - \frac{1}{2}p \end{cases}$

$\mathrm{Stab}_p[f] = \mathbb{E}_{\substack{x \text{ uniform} \\ y \sim N_p(x)}}[f(x)f(y)] = \langle f, T_p f \rangle = \sum_{S \subseteq [n]} p^{|S|} \hat{f}(S)^2$

$NS_\delta(f) = \frac{1}{2} - \frac{1}{2}\mathrm{Stab}_{1-2\delta}[f]$

$T_p f(x) = \mathbb{E}_{y \sim N_p(x)}[f(y)] = \sum_{S \subseteq [n]} p^{|S|} \hat{f}(S) \chi_S(x)$

$L \mapsto e^{(\frac{1}{2}\ln p) \cdot L} \rightarrow p \geq 0$

Laplacian of hypercube

$e^M = \sum_{i=0}^{\infty} \frac{M^i}{i!}$

T_p - used to KKL

Arrow's theorem $\rightarrow$ if $f: \{-1,1\}^n \rightarrow \{-1,1\}$ is unanimous and leads to a Condorcet winner in a 3-candidate election, f must be a dictatorship.

Proof. Probability of winner $= \frac{3}{4} - \frac{3}{4}\mathrm{Stab}_{-\frac{1}{3}}[f]$
equals 1 if $\mathrm{Stab}_{-\frac{1}{3}}[f] = -\frac{1}{3}$,
true if $\hat{f}(S) \neq 0 \rightarrow |S| = 1$.

Lemma. If $f: \{-1,1\}^n \rightarrow \{-1,1\}$ is s.t. $\hat{f}(S) \neq 0 \rightarrow |S| = 1$, then $f = \pm\chi_{\{i\}}$ for some i.

Proof. $f = \sum_{i=1}^{n} \hat{f}(\{i\}) \chi_{\{i\}}$, let x be such that $x_i = \mathrm{sign}(\hat{f}(\{i\}))$

$f(x) = \sum_{i=1}^{n} |\hat{f}(\{i\})| = 1$

Let y_n be s.t. $y_i = \mathrm{sign}(\hat{f}(\{i\}))$, except for some j

$f(y) = \sum_{i=1}^{n} |\hat{f}(\{i\})| - 2 \cdot |\hat{f}(\{j\})| = 1 \rightarrow \hat{f}(\{j\}) = 0$

$= -1 \rightarrow |\hat{f}(\{j\})| = 1.$

Problem 1 -	Rooms	1, 5
2		2, 6
3		3, 7
4		4, 8