Lecture 4

1. (O 2.31) Show that T_ρ is *positivity-preserving* for $\rho \in [-1, 1]$; i.e., $f \geq 0$ implies $T_\rho f \geq 0$. Show that T_ρ is *positivity-improving* for $\rho \in (-1, 1)$; i.e., $f \geq 0$ and $f \neq 0$ implies $T_\rho f > 0$.

   Solution:

2. (O 2.34) Show that $|T_\rho f| \leq T_\rho |f|$ pointwise for any $f : \{-1, 1\}^n \to \mathbb{R}$. Further show that for $\rho \in (-1, 1)$, equality occurs if and only if f is everywhere nonnegative or everywhere nonpositive.

   Solution:

3. (O 2.36) Show that $\mathrm{Stab}_{-\rho}[f] = -\mathrm{Stab}_\rho[f]$ if f is odd and $\mathrm{Stab}_{-\rho}[f] = \mathrm{Stab}_\rho[f]$ if f is even.

   Solution:

4. (O 2.53) The Hamming distance $\Delta(x, y) = |\{i : x_i \neq y_i\}|$ on the discrete cube $\{-1, 1\}^n$ is an example of an ℓ_1 *metric space*. For $D \geq 1$, we say that the discrete cube can be embedded into ℓ_2 with distortion D if there is a mapping $F : \{-1, 1\}^n \to \mathbb{R}^m$ for some $m \in \mathbb{N}$ such that:

   - $\|F(x) - F(y)\|_2 \geq \Delta(x, y)$ for all x, y (no contraction)
   - $\|F(x) - F(y)\|_2 \leq D\Delta(x, y)$ for all x, y (expansion at monst D)

   In this exercise you will show that the least distortion possible is $D = \sqrt{n}$.

   (a) Recalling the definition of f^{odd} from Exercise 1.8, show that for any $f : \{-1, 1\}^n \to \mathbb{R}$ we have $\|f^{\mathrm{odd}}\|_2^2 \leq \mathbf{I}[f]$ and hence

   $$\mathbb{E}_x[(f(x) - f(-x))^2] \leq \sum_{i=1}^{n} \mathbb{E}_x \left[\left(f(x) - f(x^{\oplus i}) \right)^2 \right].$$

   Solution:

   (b) Suppose $F : \{-1, 1\}^n \to \mathbb{R}^m$ and write $F(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ for functions $f_i : \{-1, 1\}^n \to \mathbb{R}$. By summing the above inequality over $i \in [m]$, show that any F with no contraction must have expansion at least $\sqrt{n}$.

   Solution:

   (c) Show that there is an embedding F achieving distortion $\sqrt{n}$.

Solution:

Lecture 5

1. (O 3.4) Prove Lemma 3.5 (listed below) by induction on n. (Hint: If one of the subfunctions $f(x_1, x_2, \ldots, x_n, \pm 1)$ is identically 0, show that the other has degree at most $k-1$.)

   Suppose $\deg(f) \leq k$ where $f : \{-1,1\}^n \to \mathbb{R}$ is not identically 0. Then $\Pr[f(x) \neq 0 \geq 2^{-k}]$.

   Solution:

2. (O 3.15) Given $f : \mathbb{F}_2^n \to \mathbb{R}$, define its *fractional sparsity* to be $\text{sparsity}(f) = |\text{supp}(f)|/2^n = \Pr_{x \in \mathbb{F}_2^n}[f(x) \neq 0]$. In this exercise, you will prove the *uncertainty princple*: If f is non-zero, then $\text{sparsity}(f) \cdot \text{sparsity}(\widehat{f}) \geq 1$:

   (a) Show that we may assume $\|f\|_1 = 1$.

   Solution:

   (b) Suppose $\mathcal{F} = \{\gamma : \widehat{f}(\gamma) \neq 0\}$. Show that $\hat{\|} f \hat{\|}_2^2 \leq |\mathcal{F}|$.

   Solution:

   (c) Suppose $\mathcal{G} = \{x : f(x) \neq 0\}$. Show that $\|f\|_2^2 \geq 2^n/|\mathcal{G}|$.
   Hint: Recall that Cauchy-Schwarz says that $\langle f, g \rangle \leq \|f\|_2 \|g\|_2$.

   Solution:

   (d) Identify all cases of equality.

   Solution:

3. (O 3.19) Let $f : \mathbb{F}_2^n \to \mathbb{R}$ be computable by a decision tree of size s and depth k Show that $-f$ and the Boolean dual $f^\dagger$ are also computable by decision trees of size s and depth k.

   Solution:

4. (O 3.34) Improve Proposition 3.31 as follows. Suppose $f : \{-1,1\}^n \to \{-1,1\}$ and $g : \{-1,1\}^n \to \mathbb{R}$ satisfy $\|f - g\|_1 \leq \varepsilon$. Pick $\theta \in [-1,1]$ uniformly at random and define $h : \{-1,1\}^n \to \{-1,1\}$ by $h(x) = \text{sgn}(g(x) - \theta)$. Show that $\mathbb{E}[\text{dist}(f,h)] \leq \varepsilon/2$.

Solution: