January 21

Thursday, January 21, 2021 1:00 PM

Boolean Functions
1. Summary
 - Influence
2. Q+A
3. Problems

Dictator$_i$ - $f(x) = x_i$

TRIBES$_{s,w}$ : OR(AND($x_1, \dots, x_w$), AND($x_{w+1}, \dots, x_{2w}$), ...)

monotone - $f(x) \le f(y)$ if $x \le y$

graph properties (connectivity)

transitive symmetric $\forall i, i'$ exists $\pi$ s.t. $\pi_i = i'$, $f(x) = f(x^\pi)$ for all $x$

unbiased - $\mathbb{E}[f] = 0$ ($\frac{1}{2}$ if range is $\{0,1\}$)

TRIBES - mean $1 - (1 - 2^{-w})^s \approx \frac{1}{2}$

$(1 - 2^{-w})^s \approx \frac{1}{2}$

$(1 - \frac{1}{x})^x \approx e$ if $x$ is large

$s = 2^w \ln 2$

$\mathrm{Inf}_i[f] = \Pr[f(x) \neq f(x^{\oplus i})]$

$x \sim y$ : $i^{th}$ bit flipped

$\mathrm{Inf}_i[\mathrm{TRIBES}_{s,w}] \sim \frac{\ln n}{n}$ $s = 2^w \ln 2$

KKL - If $f: \mathbb{F}_2^n \to \mathbb{F}_2$, exists an $i$ s.t.

$\mathrm{Inf}_i(f) = \Omega\left(\frac{\ln n}{n}\right) \mathrm{Var}[f]$ - strengthening STOC 2020

$\mathrm{Var}[f] = \mathbb{E}[f^2] - \mathbb{E}[f]$

$\mathbb{E}[f^2] = 1$, $\mathbb{E}[f] = 0$, $1 - 0 = 1$

My $D_i' f(x) = x_i D_i f(x)$ - Ryan's

eigenvalues of $D_i'$ are 1 or 0.

$D_i'^2 = D_i' = D_i^2$ $\mathrm{Inf}_i = \langle D_i f, D_i f \rangle = \langle f, D_i' f \rangle$

Thm. $D_i f = \sum_{S: i \in S} \hat{f}(S) x^{S \setminus \{i\}}$

$D_i' f = \sum_{S: i \in S} \hat{f}(S) x^S$

$\mathrm{Inf}_i[f] = \sum_{S: i \in S} \hat{f}(S)^2$

If $f$ is monotone, $\mathrm{Inf}_i(f) = \hat{f}(\{i\})$

If $f$ is transitive symmetric,

$\hat{f}(\{i\}) = \hat{f}(\{i'\})$ for all $i, i'$. - both, $\mathrm{Inf}_i[f] \le \frac{1}{\sqrt{n}}$

$\pi$ maps $i'$ to $i$ s.t. $f(x) = f(x^\pi)$

$\hat{f}(\{i\}) = \mathbb{E}[f(x) x_i] = \mathbb{E}[f(x^\pi) x_i^\pi] = \mathbb{E}[f(x) x_i^\pi] = \mathbb{E}[f(x) x_{i'}] = \hat{f}(\{i'\})$

def. Total Influence

$\mathbf{I}[f] = \sum_{i=1}^n \mathrm{Inf}_i[f] = \sum_{i=1}^n \langle f, D_i' f \rangle = \langle f, \left(\sum_{i=1}^n D_i'\right) f \rangle$

$(D_i')_{x,y} = \begin{cases} \frac{1}{2} & \text{if } x = y \\ -\frac{1}{2} & \text{if } x, y \text{ differ in } i^{th} \text{ coordinate} \\ 0 & \text{o.w.} \end{cases}$

Thm. Majority maximizes influence among monotone functions

$= \langle f, Lf \rangle \longrightarrow$ can define Influence for any graph -

$L_{x,y} = \begin{cases} \frac{n}{2} & \text{if } x = y \\ -\frac{1}{2} & \text{if } x, y \text{ differ in 1 coordinate} \\ 0 & \text{o.w.} \end{cases}$

$\frac{1}{2}$ Laplacian hypercube

Hypercube $G = (\{0,1\}^n, E)$, $E = \{(u,v) \mid u, v \text{ differ in 1 coordinate}\}$

slice = $G_{n,k}$ = { strings of length $n$ w/ $k$ 1's } differ in 2 coordinates

highdimensional expander induced subgraph of the slice

$L_{x,y} = \begin{cases} \text{deg of } x & \text{if } x = y \\ -1 & \text{if } (x,y) \in E \\ 0 & \text{o.w.} \end{cases}$

$f$ is monotone iff $D_i f \ge 0$ -

$D_i f(x) = \frac{f(x^{i \to 1}) - f(x^{i \to -1})}{2}$ = $1 \to 1, -1$; $0 \to$ same; $-1 \to -1, 1$

$f(x^{i \to 1}) \ge f(x^{i \to -1})$ if monotone

Room 1 - Problem 1