January 19

Tuesday, January 19, 2021 1:05 PM

Boolean Functions

1. Summary
   - Convolution
   - BLR test
2. Q+A
3. Problems

Announcements

- 1/21 added

Convolution - $f * g\,(x) = \mathbb{E}_y[f(x)\cdot g(y-x)]$

$= y+x$

Thm. $\widehat{f*g}(S) = \hat{f}(S)\cdot\hat{g}(S)$

H_n - rows / columns χ_S, characters

$(H_{x,y}) = -1^{\langle x,y\rangle}$, $x,y\in\{0,1\}^n$ $\langle x,y\rangle$ w/o normalizing

Let $\mathrm{diag}(g)\in\mathbb{R}^{\{0,1\}^n\times\{0,1\}^n}$, diagonal

$\mathrm{diag}(g)_{x,x} = g(x)$

Claim. $(H\cdot\mathrm{diag}(g)\cdot H)_{x,y} = \hat{g}(S)\cdot 2^n$

$i\in S$ iff exactly 1 of $x_i, y_i = 1$

Proof. $(H\,\mathrm{diag}(g)\cdot H)_{x,y} =$

$\sum_{z\in\{0,1\}^n} H_{x,z}\,\mathrm{diag}(g)_{z,z}\,H_{z,y} =$

$\sum_{z\in\{0,1\}^n} (-1)^{\langle x,z\rangle} g(z)\,(-1)^{\langle y,z\rangle}$

$= \sum_{z\in\{0,1\}^n} g(z)\,(-1)^{\langle x+y,z\rangle} = \hat{g}(S)\cdot 2^n$

$\frac{1}{2^n}H\cdot f(x) = \hat{f}(S)$ $S=\{i \mid x_i = 1\}$ $H_n^2 = 2^n\cdot I$

$\mathrm{diag}(\hat{g})\cdot\hat{f} = \frac{\mathrm{diag}(\hat{g})\cdot H_n f}{2^n} = \widehat{f*g} = \frac{H\cdot(f*g)}{2^n}$

Claim

drying

$\frac{\boxed{H\,\mathrm{diag}(\hat{g})\,H}\,f}{\boxed{2^n}\cdot 2^n} = g*f$

$\frac{H\,H(f*g)}{2^n} = f*g$

BLR - given query access to $f:\{-1,1\}^n\to$ $\mathbb{F}_2 = \{-1,1\}$

check / test if it's linear, ($f(x)+f(y)=f(x+y)$ for all x,y).

chose $x+y$ at random, check if

$f(x)+f(y)=f(x+y)$

Acceptance probability is $\frac{1}{2}+\frac{1}{2}\sum_{S\subseteq[n]}\hat{f}(S)^3$

If $\hat{f}$ is concentrated on just one S, $\sum_{S\subseteq[n]}\hat{f}(S)^3 \approx \sum_{S\subseteq[n]}\hat{f}(S)^2 = 1$

spread out $\sum_{S\subseteq[n]}\hat{f}(S)^3 << \sum_{S\subseteq[n]}\hat{f}(S)^2 = 1$

$\|f\|_p = \mathbb{E}[|f(x)|^p]^{\frac{1}{p}}$ $\to$ $f:\{-1,1\}^n\to\mathbb{R}$

Jensen's inequality if $0<p<q$,

$\mathbb{E}[|f(x)|^p]^{1/p} \le \mathbb{E}[|f(x)|^q]^{1/q}$

$\|f\|_p \le \|f\|_q$

convexity $x^{\frac{q}{p}}$

$\mathbb{E}_{x,y}[f(x)\cdot f(y)\cdot f(x+y)]$

$= f(x)\cdot\mathbb{E}_y[f(y)\cdot f(x+y)]$

$= f(x)\cdot f*f(x)$

Fourier coefficients $f\cdot g$ $\to$ convolution Fourier transform of $f*g$

$\sum \hat{f}(S)^2 = 1$, each $\hat{f}(S)^2\le 1 \to |\hat{f}(S)|\le 1$

6 breakouts

Problem 1 - Rooms 1, 4

Problem 2 - Rooms 2, 5

Problem 3 - Rooms 3, 6

Each week - 1 assigment