January 12

Tuesday, January 12, 2021 1:00 PM

Boolean Functions

1. Administrivia

2. Discrete Derivatives

Def. $f: \{0,1\}^n \to \{0,1\}$

$\{0,1\}^n \to \mathbb{R}$

Ex. AND $f(x) = \begin{cases} 1 & \text{if } x \text{ is all 1's} \\ 0 & \text{o.w.} \end{cases}$

OR $f(x) = \begin{cases} 1 & \text{if any } x_i \text{ is } 1 \\ 0 & \text{o.w.} \end{cases}$

Connectivity $f(x) = \begin{cases} 1 & \text{if } x \text{ reps a connected graph} \\ 0 & \text{o.w.} \end{cases}$

Ex. Linearity testing

f is linear if $f(x) + f(y) = f(x+y)$ for every $x, y \in \{0,1\}^n$

$(1+1=0)$

How can we decide if f is linear?

Idea - test every x, y pair - quadratic

Idea - test random pair - constant

Ex. How to extrapolate?

Know the value of f on $S \subseteq \{0,1\}^n$

how to extrapolate to all inputs?

(learning theory)

Ex. Voting schemes

Real-valued functions

$f: \mathbb{R}^n \to \mathbb{R}$

Def. Derivative

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}$$

$$e_i = (\underbrace{0, \dots, 0}_{i-1}, \underset{i}{1}, \underbrace{0, \dots, 0}_{n-i})$$

Def. Discrete Derivative

$x = (x_1, x_2, \dots, x_n)$

$x^{\oplus i} = (x_1, \dots, x_{i-1}, 1 - x_i, x_{i+1}, \dots, x_n)$

$$D_i f(x) = \frac{f(x) - f(x^{\oplus i})}{2}$$

(Video does something diff)

Facts. Discrete derivative is linear

$D_i(f+g) = D_i f + D_i g \qquad D_i \cdot (cf) = c \cdot D_i f$

- Set of boolean functions forms a finite vector space. (can think of boolean functions $\mathbb{R}^{\{0,1\}^n}$

$D_i \in \mathbb{R}^{\{0,1\}^n \times \{0,1\}^n}$

Thm. D_i are simultaneously diagonalizable (share a basis of eigenvectors).

$n = 2 \qquad \{0,1\}^n = 00 \; 01 \; 10 \; 11$

$$D_1 f(00) = \frac{f(00) - f(10)}{2}$$

$$D_1 f(11) = \frac{f(11) - f(01)}{2}$$

$$\begin{array}{cc} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ D_1 = & \begin{bmatrix} \frac{1}{2} & & -\frac{1}{2} & \\ & \frac{1}{2} & & -\frac{1}{2} \\ -\frac{1}{2} & & \frac{1}{2} & \\ & -\frac{1}{2} & & \frac{1}{2} \end{bmatrix} \end{array} \begin{bmatrix} f(00) \\ f(01) \\ f(10) \\ f(11) \end{bmatrix} \qquad \frac{f(00) - f(10)}{2} \quad \frac{f(11) - f(01)}{2}$$

$\{0,1\}^n$ - abelian group

$1+1=0 \qquad x = (0,1,0,1), \; y = (0,1,1,1)$

$x+y = (0,0,1,0)$

Def. One-dimensional representation $f: \{0,1\}^n \to \{-1,1\}$

s.t. $f(x+y) = f(x) \cdot f(y) \qquad f(x+x) = f(0) = 1 = f(x)^2$

$f(0, \dots, 0) = 1$

Ex. $f_1(x) = 1$ for all x - is a 1-D rep

$f_2(x) = (-1)^{x_1}$ - $f_2(x) = \begin{cases} 1 & \text{if } x_1 = 0 \\ -1 & \text{if } x_1 = 1 \end{cases}$

$f_2(0, \dots, 0) = (-1)^0 = 1$

x_1	y_1	$x_1 + y_1$	$f(x)$	$f(y)$	$f(x+y)$
0	0	0	1	1	1
1	0	1	-1	1	-1
0	1	1	1	-1	-1
1	1	0	-1	-1	1

$\to$ f_2 corresponds to $y = (1, 0, 0, \dots, 0)$

Def. $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ (Inner product)

$x, y \in \{0,1\}^n$

Claim. For every $y \in \{0,1\}^n$, $f_y: \{0,1\}^n \to \mathbb{R}$ defined by $f_y(x) = (-1)^{\langle x, y \rangle}$ is a 1-D rep.

Proof. $f(0, \dots, 0) = (-1)^{\langle 0, y \rangle} = (-1)^0 = 1$

$f(x+z) = (-1)^{\langle x+z, y \rangle} = (-1)^{\langle x, y \rangle + \langle z, y \rangle}$

$(\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle)$

$= (-1)^{\langle x, y \rangle} \cdot (-1)^{\langle z, y \rangle}$

$= f(x) \cdot f(z)$

Claim. Every 1-D rep can be expressed in this way.

Proof. Let f be a 1-D rep.

Let $y \in \{0,1\}^n$ be defined by $y_i = \begin{cases} 0 & \text{if } f(e_i) = 1 \\ 1 & \text{if } f(e_i) = -1 \end{cases}$

$$e_i = (\underbrace{0, \dots, 0}_{i-1}, \underset{i}{1}, 0, \dots, 0)$$

$$f(x) = \left(\prod_{i : x_i = 1} f(e_i) \right) \cdot f(0) = \prod_{\substack{i : x_i = 1 \\ y_i = 1}} -1 = (-1)^{\langle x, y \rangle}$$

$\downarrow$

$f(x+y) = f(x) \cdot f(y)$

Fact. $y_1 \neq y_2$ then functions $f_{y_1}(x) = (-1)^{\langle x, y_1 \rangle}$, $f_{y_2}(x) = (-1)^{\langle x, y_2 \rangle}$

They are orthogonal, $\sum_{x \in \{0,1\}^n} f_{y_1}(x) \cdot f_{y}(x) = 0$

Proof. In video for 1/14

Corollary - 1-D reps. form an orthogonal basis for $\mathbb{R}^{\{0,1\}^n}$

$$D_i f(x) = \frac{f(x) - f(x^{\oplus i})}{2}$$

Thm. D_i share an orthogonal basis of eigenvectors.

Proof. $f_y(x) = (-1)^{\langle x, y \rangle}$

$$(D_i f_y)(x) = \frac{f_y(x) - f_y(x^{\oplus i})}{2}$$

Let $a = \begin{cases} 1 & \text{if } y_i = 1 \\ 0 & \text{if } y_i = 0 \end{cases}$

$$= \frac{(-1)^{\langle x, y \rangle} - (-1)^{\langle x^{\oplus i}, y \rangle}}{2}$$

$$= \frac{(-1)^{\langle x, y \rangle} - (-1)^{\langle x, y \rangle} \cdot (-1)^a}{2} = (-1)^{\langle x, y \rangle} \frac{(1 - (-1)^{y_i})}{2} = f_y(x) \cdot \frac{(1 - (-1)^{y_i})}{2}$$

eigenvalue is 1 if $y_i = 1$

0 if $y_i = 0$

Def. Hadamard matrix

$H_n \in \mathbb{R}^{\{0,1\}^n \times \{0,1\}^n}$

$H_n(x, y) = (-1)^{\langle x, y \rangle}$ - columns are 1-D reps

rows are 1-D reps

$$H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix} \qquad H_0 = [1]$$

Def. Fourier transform of f ($f \in \mathbb{R}^{\{0,1\}^n}$, also a function $f: \{0,1\}^n \to \mathbb{R}$)

$$\frac{1}{2^n}(H_n f) = \hat{f}$$

Fact. $H_n^2 = 2^n \cdot I$

(rows are orthogonal).