January 14

Thursday, January 14, 2021 1:45 PM

Boolean Functions

1. Summary of Lecture 1
   - Fourier Expansion
   - Basic Formulas
2. Q+A
3. Problems

Lecture 1

Vector space of functions $f:\{-1,1\}^n \to \mathbb{R}$, $\mathbb{R}^{\{0,1\}^n}$

Inner product $\langle f, g\rangle = \sum_{x\in\{-1,1\}^n} \frac{f(x)g(x)}{2^n}$

Orthonormal basis χ_S $S \subseteq [n] = \{1,\dots,n\}$

$\chi_S(x) = \prod_{i\in S} x_i$

Fourier coefficient $\hat{f}(S) = \langle f, \chi_S\rangle$

$\langle \chi_S, \chi_S\rangle = 1$ for all S

Fact. If $f, g: \{-1,1\}^n \to \{-1,1\}$

$\langle f, g\rangle = 1 - 2\Pr[f(x) \neq g(x)]$

(Plancherel) $\langle f, g\rangle = \sum_{S\subseteq[n]} \hat{f}(S)\hat{g}(S)$

(Parseval) $\|f\|_2^2 = \sum_{S\subseteq[n]} \hat{f}(S)^2$

Change of basis

$\mathbb{E}[f] = \langle f, \mathbf{1}\rangle = \hat{f}(\emptyset)$

all 1s vector

$\mathrm{Var}[f] = \mathbb{E}[f^2] - \mathbb{E}[f]^2 = \langle f, f\rangle - \hat{f}(\emptyset)^2$
$= \|f\|_2^2 - \hat{f}(\emptyset)^2 = \sum_{S\subseteq[n],\, S\neq\emptyset} \hat{f}(S)^2$

Ex. $x_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $x_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$x_i = \begin{bmatrix} \left.\begin{matrix} -1 \\ -1 \\ \vdots \\ -1 \end{matrix}\right\} i-1 \\ i-1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

also orthogonal

also satisfy

$\mathbb{E}[F] = \hat{f}(0)$

$\mathrm{Var}[F] = \sum_{S\neq 0} \hat{f}(S)^2$

χ_S, $\prod_{i\in S} x^i$, x^S - all same

$\downarrow$

Comes representation theory from

$\max_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\chi_\emptyset} + \frac{1}{2}\,\chi_{\{1\}},$

$\chi_{\{1,2\}} = x_1 x_2$, $-\frac{1}{2}\underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}_{\chi_{\{1,2\}}}$

$\mathbb{F}_2$ - $\{0,1\}$ addition
$\{-1,1\}$ multiplication

$\underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}_{\chi_{\{1,2\}}}$

00	1 1
01	1 -1
10	-1 1
11	-1 -1

$\underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\chi_{\{1\}}}$

$f = \sum_{S\subseteq[n]} \langle f, \chi_S\rangle \cdot \chi_S$

$\langle f, \chi_\emptyset\rangle = \sum_x f(x)\chi_\emptyset(x)\cdot\frac{1}{2^n}$
$= \sum_x f(x)\cdot\frac{1}{2^n}$

Breakout rooms

Rooms - 1 Problem 1
2 Problem 2
3 Problem 3
4 Problem 4

Assignment on Canvas