Lecture 2

1. (O 1.23) In this exercise, you will prove some basic facts about "distances" between probability distributions. Let ϕ and ψ be probability densities on $\mathbb{F}_2^n$.

   Recall that
   $$\|f\|_p = \mathbb{E}[|f(x)|^p]^{1/p}$$
   and that Jensen's inequality states that if $0 < p < q$, then
   $$\mathbb{E}[|f(x)|^p]^{1/p} \leq \mathbb{E}[|f(x)|^q]^{1/q}.$$

   (a) Show that the *total variation distance* between ϕ and ψ, defined by
   $$d_{\mathrm{TV}}(\phi, \psi) = \max_{A \subseteq \mathbb{F}_2^n} \{|\mathrm{Pr}_{y \sim \phi}[y \in A] - \mathrm{Pr}_{y \sim \psi}[y \in A]|\}$$
   is equal to $\frac{1}{2}\|\phi - \psi\|_1$.

   Solution:

   (b) Show that the *collision probability* of ϕ, defined to be
   $$\Pr_{\substack{y, y' \sim \phi \\ \text{independently}}}[y = y']$$
   is equal to $\|\phi\|_2^2 / 2^n$.

   Solution:

   (c) The χ^2-*distance of* ϕ *from* ψ is defined by
   $$d_{\chi^2}(\phi, \psi) = \mathbb{E}_{y \sim \psi}\left[\left(\frac{\phi(y)}{\psi(y)} - 1\right)^2\right],$$
   assuming ψ has full support. Show that the χ2-distance of ϕ from uniform is equal to $\mathrm{Var}[\phi]$.

   Solution:

   (d) Show that the total variation distance of ϕ from uniform is at most $\frac{1}{2}\sqrt{\mathrm{Var}[\phi]}$.

   Solution:

2. Let $f : \mathbb{F}_2^n \to \mathbb{R}$. Construct functions $g_1, g_2 : \mathbb{F}_2^n \to \mathbb{R}$ such that

   (a) $f * g_1 = D_i f$ where $D_i f(x) = (f(x) - f(x^{\oplus i}))/2$

   Solution:

   (b) $f * g_2 = T_\rho f$ where $T_\rho f = \sum_{S \subset [n]} \rho^{|S|} \widehat{f}(S) \chi_S$.

Solution:

3. (O 1.29)

   (a) We call $f : \mathbb{F}_2^n \to \mathbb{F}_2$ an *affine* function if $f(x) = a \cdot x + b$ for some $a \in \mathbb{F}_2^n, b \in \mathbb{F}_2$. Show that f is affine if and only if $f(x) + f(y) + f(x) = f(x + y + z)$ for all $x, y, z \in \mathbb{F}_2^n$.

   Solution:

   (b) Let $f : \mathbb{F}_2^n \to \mathbb{R}$. Suppose we choose $x, y, z \sim \mathbb{F}_2^n$ independently and uniformly. Show that $\mathbb{E}[f(x)f(y)f(z)f(x + y + z)] = \sum_S \hat{f}(S)^4$.

   Solution:

   (c) Give a 4-query test for a function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ with the following property. If the test accepts with probability $1-\varepsilon$ then f is ε-close to being affine. All four query inputs should have the uniform distribution on $\mathbb{F}_2^n$ (but of course need not be independent).

   Solution:

   (d) Give an alternate 4-query test for being affine in which three of the query inputs are uniformly distributed and the fourth is not random. (Hint: Show that f is affine if and only if $f(x) + f(y) + f(0) = f(x + y)$ for all $x, y \in \mathbb{F}_2^n$.)

   Solution:

Lecture 3

1. (O 2.3) Prove *May's Theorem*:

   (a) Show that $f : \{-1, 1\}^n \to \{-1, 1\}$ is symmetric and monotone if and only if it can be expressed as a weighted majority with $a_1 = a_2 = \cdots = a_n = 1$. (That is, $f(x) = \text{sign}(a_0 + x_1 + x_2 + \cdots + x_n)$ for some a_0)

   Solution:

   (b) Suppose $f : \{-1, 1\}^n \to \{-1, 1\}$ is symmetric, monotone, and odd. Show that n must be odd, and that $f = \text{Maj}_n$.

   Solution:

2. (O 2.19) Suppose $f, g : \{-1, 1\}^n \to \mathbb{R}$ have the property that f does not depend on the ith coordinate and g does not depend on the jth coordinate ($i \neq j$). Show that $\mathbb{E}[x_i x_j f(x)g(x)] = \mathbb{E}[D_j f(x) D_i g(x)]$.

   Here, $D_i f(x) = \frac{f(x^{i\mapsto 1}) - f(x^{i\mapsto -1})}{2}$.

Solution:

3. (Based on O 2.15) Define $E_i f(x) = \frac{f(x^{i\mapsto 1}) + f(x^{i\mapsto -1})}{2}$. Prove that $f = x_i D_i f + E_i f$, and give the Fourier coefficients of $E_i f$ in terms of the Fourier coefficients of f.

Solution:

4. (Based on O 2.27) Which functions $f : \{-1,1\}^n \to \{0,1\}$ such that $|\{x : f(x) = 1\}| = 3$ maximize $\mathbf{I}[f]$?

Solution: