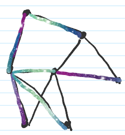
November 6

Friday, November 6, 2020 4:07 PM

# Comp Sci 212
## 1. Minimum Spanning Trees

Def. A spanning tree of a graph $G=(V,E)$ is a subgraph $G'=(V,E')$ such that $G'$ is a tree. ($G'$ is connected, acyclic, $|E|=|V|-1$)



- Minimal connected subgraph w/ all vertices.

Only connected graphs have spanning trees.

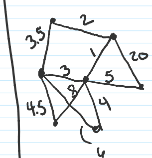
### Weighted graph

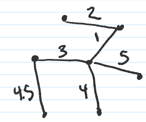
$G=(V,E,w)$ $w: E \to \mathbb{R}$

Transportation networks
Internet networks
Social networks

weights can represent costs, strength

Problem - How to find a spanning tree whose sum of edges is as small as possible? (Minimum spanning tree).




Graphs can have multiple MST's, (all edges w/ same weight)

### Kruskal's algorithm

1. Sort the edges, from smallest to largest
2. Consider each edge from smallest to largest
3. Add an edge if it doesn't create a cycle.

Thm. Kruskal's algorithm returns a spanning tree. (T - output)

Proof. T is acyclic, (because of step 3), so just need to prove it's connected.
Use contradiction, assume $v_1$ & $v_n$ are connected in G, but not in T. Let $(v_1, v_2, \ldots, v_{n-1}, v_n)$ be a path from $v_1$ to $v_n$ in G. Then must exist $v_i, v_{i+1}$ not connected in T.

Kruskal's algorithm does not add edge $\{v_i, v_{i+1}\}$. This must be adding would create a cycle. This implies $v_i, v_{i+1}$ are already connected in T, contradiction.

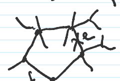
Thm. Kruskal outputs a MST → assume edge weights are distinct

Proof. T be tree output by Kruskal

Assume T is not a MST, for sake of contradiction. Let e be first edge add T, but not in some MST. (all edges in T w/ smaller weight are in M)

Let M be that MST. Adding e to M creates a cycle. The cycle must have an edge f, not in T. Let M' be M, after removing f add e. M' is a spanning tree (exercise check connected, $|E|=|V|-1$)

M w/ f




We'll show M' has smaller total weight than M.

→ weight of M' = weight M - weight (f) + weight (e)

Have a contradiction if $w(f) > w(e)$.

Show that $w(f) < w(e)$ is not possible, so contradiction occurs.

Assume $w(f) < w(e)$. f is not in T, Kruskal's algorithm chose not to add f. Adding f would have created a cycle in T. But, all the other edges are in M, f is in M, so M has a cycle, contradicting M is a MST. □

Note: If $w(f) = w(e)$ is possible, repeat on M', keep repeating, have contradiction, or obtain T.