November 9

Monday, November 9, 2020 4:05 PM

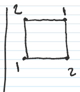
Comp Sci 212
1. Graph Coloring
2. Bipartite Graphs

Def. A graph $G=(V,E)$ is K-colorable if there exists a function $\underbrace{f: V \to}_{\text{coloring}} \underbrace{[K]}_{\text{colors}}$ s.t. if $\{u,v\} \in E$, then $f(u) \neq f(v)$.

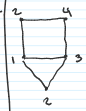
Def. The chromatic number of a graph $\chi(G)$ is the smallest K s.t. G is K-colorable

- If G is K-colorable, $\chi(G) \leq K$.
- $\chi(G) \leq |V|$

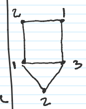
1, 2, 3, 4 - colors



2-colorable $\chi(G)=2$



4-colorable



3-colorable $\chi(G)=3$

Cycle. $V=[n]$ $E=\{\{1,2\},\{2,3\},\dots,\{n-1,n\},\{1,n\}\}$

Thm. If $G=([n],E)$ is a cycle and n is even, $\chi(G)=2$.

Proof. G has edges, so $\chi(G)>1$.

Let $f:[n]\to\{1,2\}$

$$f(v)=\begin{cases}1 & \text{if } v \text{ is odd}\\ 2 & \text{if } v \text{ is even.}\end{cases}$$

Then if $\{u,v\}\in E$, either $v=u+1$, and $f(u)\neq f(v)$, or $u=1$, $v=n$, so $f(u)=1$, $f(n)=2$.

$\square$

Thm. If $G=([n],E)$ is a cycle, n is odd, $\chi(G)=3$.

Proof. Prove that $\chi(G)>2$ by contradiction.

Assume G is 2-colorable $f:[n]\to[2]$.

Then, $f(2)\neq f(1)$, $f(3)\neq f(2)$, so $f(1)=f(3)$. In general, $f(i)\neq f(i+1)$, $f(i+1)\neq f(i+2)$, $f(i)=f(i+2)$ $(i \leq n-2)$

$f(1)=f(3)=f(5)=\dots=f(n)$. f is not a valid 2-coloring, $\{1,n\}\in E$.

G is 3-coloring $f:[n]\to[3]$.

$$f(v)=\begin{cases}1 & \text{if } v \text{ is odd and } v\neq n\\ 2 & \text{if } v \text{ is even}\\ 3 & \text{if } v \text{ is } n\end{cases}$$

$\square$

Complete graph (all possible edges) edge for every pair of vertices

Thm. If $G=([n],E)$ is the complete graph, $\chi(G)=n$.

Proof. All colors must be different from each other, so $\chi(G)\geq n$. $\chi(G)\leq n$, $f:[n]\to[n]$, $f(v)=v$, valid n-coloring.

Thm. If $G=(V,E)$ is a graph where all vertices have degree K or less, G is $(K+1)$-colorable.
↳ $\deg(v)\leq k$ for all v

each vertex is connected to at most K other vertices.

Proof. Use induction.

Base case - $|V|=1$, it's 1-colorable. → $|V|=n$.

Inductive - Let $v\in V$, $S=\{e \mid e\in E, v\in e\}$
↳ set of edges that contain v.

Assume true if less than n vertices

Let $G'=(V\setminus\{v\}, E\setminus S)$

G' is $(K+1)$-colorable by inductive hypothesis. Use same coloring for G, color v anything diff. from neighbors. This is possible because v has at most K neighbors.

$\square$