November 2

Monday, November 2, 2020 4:01 PM

Comp Sci 212
1. Degree
2. Connected components

Announcements
- Homework 5 due Friday

Def. (Graph) Vertex set V, edge set $E \subseteq V \times V$
$G = (V, E)$

element of E
subsets of size 2.

Def. Degree of a vertex $v \in V$, $\underline{deg(v)}$, $d(v)$, d_v
$|\{e \mid e \in E, v \in e\}|$, # of edges that contain v.

Def. neighbor of v is a vertex s.t. $\{v, u\} \in E$.

Thm. $\sum_{v \in V} deg(v) = 2|E|$ (handshake lemma)

Prof. Each edge contributes 1 to the degree of two vertices.

$\square$

Prof. S = set of subsets of V of size 2.

$f: S \to \mathbb{R}$, $f(\{v, u\}) = 1$ if $\{u, v\} \in E$
$f(\{u, v\}) = 0$ o.w.

$$\sum_{v \in V} deg(v) = \sum_{v \in V} \sum_{\substack{u \in V \\ v \neq u}} f(\{v, u\})$$

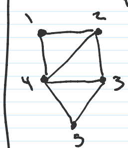
$$= \sum_{\substack{(v,u) \in V \times V \\ v \neq u}} f(\{v, u\}) = 2 \cdot |E|$$

$\square$

Corollary Number of vertices w/ odd degree is even.

Prof. Use Contradiction. If # of vertices w/ odd degree is odd, $\sum deg(v)$ is odd, contradicts $= 2 \cdot |E|$.

Def. (path) A path is a sequence of vertices $(v_1, v_2, \dots, v_n)$, $\{v_i, v_{i+1}\} \in E$ for all i.
repeats are okay, sometimes not okay



1-2-4-5
1-4-3

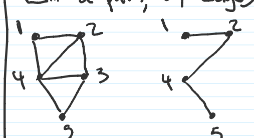
Def. u + v are connected if there is a path from u to v.

Def. A graph is connected if all pairs of vertices are connected.

Ex. not connected

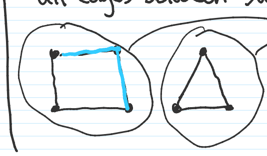
Def. Subgraph of $G = (V, E)$ is $G' = (V', E')$
$V' \subseteq V$, $E' \subseteq E$, if $\{u, v\} \in E'$, then $u, v \in V'$

Ex. a path, w/ edges



graph sub-graph

Def. Connected component - a subgraph contain a vertex v, all vertices connected to v, all edges between such vertices. (present in original graph)



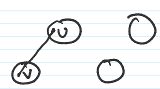
connected component

(connected graph have 1 connected component.

Thm. Every graph $G = (V, E)$ has at least $|V| - |E|$ connected components.

Proof. Use induction on # of edges.
Base case - $|E| = 0$, $|V|$ cc's (each vertex in its own cc)
Inductive step - Assume is true if $|E| \le k$.
If $|E| = k+1$, consider graph $G' = (V, E \setminus \{e\})$ for some $e \in E$. G' has at least $|V| - k$ cc's by inductive hypothesis.
Either $e = \{v, u\}$, v + u are in same cc - 1
different cc's - 2

Case 1 - # of cc is same, # of cc $\ge |V| - k \ge |V| - k - 1$.
Case 2 - # of cc's decreases by 1,
#cc's in G = # of cc's in G' - 1
$\ge |V| - k - 1$

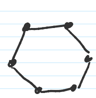


$\square$

Corollary. In a connected graph, $|E| \ge |V| - 1$.

Proof. In a connected graph, $|V| - |E| \le 1$.

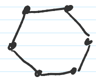
Def. A graph is acyclic if no subgraph is a cycle.



- cycles

Corollary If a graph is acyclic, and $|E| = |V| - 1$, it is connected.

a cycle.



– cycles

Corollary. If a graph is acyclic, it has exactly $|V|-|E|$ cc's.

Proof. Need to prove Case 1 (from Thm.) never happens.

Assume case 1 happens, for sake of contradiction. $e=\{u,v\}$, exist a path $(v, v_1, v_2, \dots, u)$ b/c u & v are in same cc. Adding e, creates a cycle, contradicting G being acyclic.