October 30

Friday, October 30, 2020 4:02 PM

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Comp Sci 212

1. Chernoff bounds
2. Graph theory

Chernoff bounds (law of large number - CS version)

Thm. If $R_1, \dots, R_n$ independent random variables, $R_i : S \to [-1, 1]$, $\mathbb{E}[R_i] = 0$. Any v

$$\Pr[R_1 + \dots + R_n \geq v\sqrt{n}] \leq e^{-\frac{v^2}{4e}}$$

$e = 2.7183$ ... Gaussian, $e^{-v^2/c}$

(Chebyshev) $\leq \frac{\mathrm{Var}[R_1 + \dots + R_n]}{v^2 n} \leq \frac{1}{v^2}$

$$\mathrm{Var}[R_1 + \dots + R_n] = \sum_{i=1}^{n} \mathrm{Var}[R_i] = \sum_{i=1}^{n} \mathbb{E}[R_i^2] \leq n$$

$\frac{1}{v^2} = \omega(e^{-v^2/4e})$

Proof.

$\Pr[R_1 + \dots + R_n \geq v\sqrt{n}] =$ → t to be picked later

$\Pr[e^{t(R_1 + \dots + R_n)} \geq e^{tv\sqrt{n}}] \leq$ $0 \leq t \leq 1$

$\frac{\mathbb{E}[e^{t(R_1 + \dots + R_n)}]}{e^{tv\sqrt{n}}}$ ← upper

(by Markov's inequality) $\leq \frac{(e^{t^2 e})^n}{e^{tv\sqrt{n}}} = \frac{e^{t^2 n e}}{e^{tv\sqrt{n}}} = e^{\frac{v^2}{4e} - \frac{v^2}{2e}} = e^{\frac{-v^2}{4e}}$

$\mathbb{E}[e^{t(R_1 + \dots + R_n)}] = \mathbb{E}[e^{tR_1}]\mathbb{E}[e^{tR_2}] \dots$

Moment generating function

if $R_1, R_2, \dots$ independent, then $e^{tR_1}, e^{tR_2}, \dots$ independent

$t = \frac{v}{2e\sqrt{n}}$ $= e^{t^2 n e - tv\sqrt{n}}$

$$\mathbb{E}[e^{tR_i}] = \mathbb{E}\left[1 + tR_i + \frac{t^2 R_i^2}{2} + \frac{t^3 R_i^3}{3!} + \dots\right] \quad \text{(Taylor series)}$$

$$= 1 + \mathbb{E}[tR_i] + \sum_{i=2}^{\infty} t^i \frac{\mathbb{E}[R_i^i]}{i!} \to \leq 1$$

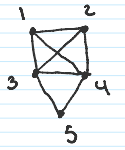
$$\leq 1 + \sum_{i=2}^{\infty} \frac{t^i}{i!} \leq 1 + t^2 \sum_{i=2}^{\infty} \frac{1}{i!} \leq 1 + t^2 e \leq e^{t^2 e}$$

$t \leq 1$, $t^i \leq t^2$ $(1 + x \leq e^x)$

Graphs

Def. (Graph) Vertex set V, Edge set $E \subseteq V \times V$

$G = (V, E)$



$V = \{1, 2, 3, 4, 5\}$

$E = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (3,5), (4,5), \dots\}$

(undirected) (and their reverse)

Graphs can represent

Social networks, V = people E = friendships

Transportation networks V = stops E = routes

Applications V = people, jobs E = applications (union)

Directed graph - $(v, v) \in E$ does not imply $(v, v) \in E$

(Instagram) (arrows)

Undirected graph - $(v, v) \in E$ implies $(v, v) \in E$

(Facebook) (no arrows)

Weighted graph, weight function, $w : E \to \mathbb{R}$



(length of route strength of friendship)

Multigraph - E multi-set (repeated elements)

Def. self-loop an edge of the form (v, v).



Def. Simple graph an undirected graph with no self-loops, not weighted, not a multi-graph E can be subset of set of subsets of size 2.

↓ edges are subsets rather than pairs

Complete graph



E = set of subsets of size 2

$|E| = \binom{n}{2}$

Thm.

of size 2.

edges are subsets rather than pairs



$|E| = \binom{n}{2}$

Tree



(talk about later)

Common graphs, $V = [n]$ for all

path

$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \ldots, \{n-1, n\}\}$.

cycle 

$E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{1, n\}\}$

star 

$E = \{\{1,2\}, \{1,3\}, \ldots, \{1, n\}\}$