October 23

Friday, October 23, 2020 4:01 PM

Comp Sci 212
1. Conditional Expectation
2. Binomial Distribution
3. Expectation of Product

Announcements
- Midterm grades out

Expected Value

$$\mathbb{E}[X] = \sum_{w \in S} X(w) \Pr[w]$$

Def. (Conditional Expectation) X r.v. A event

$$\mathbb{E}[X \mid A] = \sum_{w \in A} X(w) \cdot \Pr[w \mid A] = \frac{1}{\Pr[A]} \sum_{w \in A} X(w) \Pr[w]$$

Law of total expectation, X random variable, $A_1, \dots, A_n$ disjoint events, $A_1 \cup A_2 \cup \dots \cup A_n = S$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \cdot \Pr[A_i].$$

Mean time until failure

Each day your computer fail w.p. p, how long does your computer last on average?

$S = \{1, 2, 3, \dots\}$ (sample space)

$\Pr[i] = (1-p)^{i-1} \cdot p \rightarrow \sum_{i=1}^{\infty} p \cdot (1-p)^{i-1} = p \sum_{i=0}^{\infty} (1-p)^i = p \cdot \frac{1}{1-(1-p)} = 1$

$R: S \to \mathbb{R}$, $R(s) = s$

$\mathbb{E}[R] = \sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p \cdot i$

$\downarrow$ can compute

Let $A_1 = \{1\}$, $A_2 = S \setminus \{1\}$

$\Pr[A_1] = p$, $\Pr[A_2] = 1-p$.

By law of total expectation,

$$\mathbb{E}[R] = \mathbb{E}[R \mid A_1] \cdot \Pr[A_1] + \mathbb{E}[R \mid A_2] \Pr[A_2]$$

$= 1 \cdot p + (1-p)(\mathbb{E}[R] + 1)$

$= p + \underline{(1-p)(\mathbb{E}[R]+1)}$

$\mathbb{E}[R] = p + \underline{1-p} + \underline{(1-p)\mathbb{E}[R]}$

$\mathbb{E}[R] = 1 + (1-p)\mathbb{E}[R]$

$p\mathbb{E}[R] = 1$

$\mathbb{E}[R] = \frac{1}{p}$

$\mathbb{E}[R \mid A_2] = \frac{1}{(1-p)} \sum_{i=2}^{\infty} (1-p)^{i-1} p \cdot i = \sum_{i=2}^{\infty} (1-p)^{i-2} \cdot p \cdot i$

$= \sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p \cdot (i+1)$

$= \mathbb{E}[R+1] = \mathbb{E}[R] + 1$

$(R+1(s) = s+1)$

Binomial distribution.

$S = \{0,1\}^n$ $\Pr[s] = p^{\# \text{ of 1's in } s} (1-p)^{\# \text{ of 0's in } s}$

$B(n,p)$ - name of dist $\hookrightarrow \sum_{s \in \{0,1\}^n} p^{\# \text{ of 1's in } s} (1-p)^{\# \text{ 0's in } s} = (p + (1-p))^n = 1$

$R: S \to \mathbb{R}$ $R = \#$ of 1's.

$p = \frac{1}{2}$, same uniform dist (coins), $\Pr[s] = \frac{1}{2^n}$.

$$\overset{*}{\sum_{s \in \{0,1\}^n}} \Pr[s] = \sum_{i=0}^{n} (\# \text{ of sequences w/ } i \text{ 1's}) p^i (1-p)^{n-i}$$

$= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$

$= (p + (1-p))^n$ (binomial theorem)

$= 1$

$\mathbb{E}[R] = \sum_{s \in \{0,1\}^n} \Pr[s] \cdot (\# \text{ of 1's in } s)$

$= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \cdot i$ (Homework 4 P1)

E_i = event where i^{th} element is 1

$= \{s \mid s \in \{0,1\}^n, s_i = 1\}$

$\mathbb{1}_{E_i}(s) = \begin{cases} 1 & \text{if } s \in E_i \\ 0 & \text{o.w.} \end{cases}$ $\mathbb{1}_{E_i}: S \to \{0,1\}$.

$R(s) = \sum_{i=1}^{n} \mathbb{1}_{E_i}(s)$ $\mathbb{E}[R] = \sum_{i=1}^{n} \Pr[E_i] = n \cdot p$

$\Pr[E_i] = \sum_{\substack{s \in \{0,1\}^n \\ s_i = 1}} p^{\# \text{ of 1's in } s} (1-p)^{\# \text{ of 0's in } s}$

$= p \sum_{\substack{s \in \{0,1\}^n \\ s_i = 1}} p^{\# \text{ of 1's in } s - 1} (1-p)^{\# \text{ of 0's in } s}$ (similar to *)

$= p \cdot 1$

Def. Two r.v.'s A, B independent if

$\Pr[A = x_1] \cdot \Pr[B = x_2] = \Pr[A = x_1 \text{ and } B = x_2]$

for all x_1, x_2.

Fact. $\mathbb{E}[X] = \sum_{r \in \text{Range}(X)} r \cdot \Pr[X = r]$.

Proof. $\mathbb{E}[X] = \sum_{w \in S} X(w) \Pr[w]$

$= \sum_{r \in \text{Range}(X)} \sum_{w: X(w) = r} \Pr[w] \cdot X(w)$

$= \sum_{r \in \text{Range}(X)} r \Pr[X = r]$ □

Expected Value of Product

If R_1, R_2 are independent random variables,

$\mathbb{E}[R_1]\mathbb{E}[R_2] = \mathbb{E}[R_1 \cdot R_2]$

$(R_1 \cdot R_2(s) = R_1(s) \cdot R_2(s))$

(counter-example dependent $R_1 = R_2$, die, $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$)

Proof. $\mathbb{E}[R_1 \cdot R_2] = \sum_{r \in \text{Range}(R_1 \cdot R_2)} r \cdot \Pr[R_1 \cdot R_2 = r] = \sum_{r \in \text{Range}(R_1 \cdot R_2)} r \sum_{\substack{r_1 \in \text{Range}(r_1) \\ r_2 \in \text{Range}(r_2) \\ r = r_1 \cdot r_2}} \Pr[R_1 = r_1 \text{ and } R_2 = r_2]$

$= \sum_{\substack{r_1 \in \text{Range}(R_1) \\ r_2 \in \text{Range}(R_2)}} r_1 \cdot r_2 \Pr[R_1 = r_1 \text{ and } R_2 = r_2]$

$= \sum_{r_1 \in \text{Range}(R_1)} r_1 \cdot r_2 \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2]$

$$= \sum_{r \in Range(X)} r \Pr[X=r]$$

$\square$

$$= \sum_{r_i \in Range(R_i)} r_1 \cdot r_2 \Pr[R_1=r_1] \cdot \Pr[R_2=r_2]$$

$$= \left(\sum_{r_1 \in Range(r_1)} r_1 \Pr[R_1=r_1]\right)\left(\sum_{r_2 \in Range(r_2)} r_2 \Pr[R_2=r_2]\right)$$

$$= \mathbb{E}[R_1]\mathbb{E}[R_2]$$