October 21

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Comp Sci 212

1. Expected Value

Announcements
- Homework 4 out

Def. (Expected Value) X random variable
Sample space S, Prob dist $\Pr$
$X: S \to \mathbb{R}$

$$\underbrace{\mathbb{E}}_{\text{expected value}}[\underbrace{X}_{\text{random variable}}] = \sum_{\omega \in S} X(\omega)\Pr[\omega]$$

(like mean or average)

Ex. 6-sided die $S = \{1,2,3,4,5,6\}$, uniform dist.

$X: S \to \mathbb{R}$, $X(\omega) = \omega$

$$\mathbb{E}[X] = \sum_{\omega \in S} X(\omega)\Pr[\omega] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = \frac{7}{2}$$

Ex.
dist. $\Pr[1] = \frac{1}{2}$, $\Pr[\omega] = \frac{1}{10}$ for all otherw

$$\mathbb{E}[X] = \frac{1}{2}\cdot 1 + \frac{1}{10}\cdot 2 + \frac{1}{10}\cdot 3 + \frac{1}{10}\cdot 4 + \frac{1}{10}\cdot 5 + \frac{1}{10}\cdot 6 = \frac{5}{2}$$

Ex. $Y: S \to \mathbb{R}$ $Y(\omega) = \frac{1}{X(\omega)} = \frac{1}{\omega}$ $Y = \frac{1}{X}$ (Uniform)

$$\mathbb{E}[Y] = \sum_{\omega \in S} Y(\omega)\Pr[\omega] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot\frac{1}{2} + \frac{1}{6}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{1}{4} + \frac{1}{6}\cdot\frac{1}{5} + \frac{1}{6}\cdot\frac{1}{6} = \frac{49}{120}$$

$$\mathbb{E}[Y] \neq \frac{1}{\mathbb{E}[X]}$$

$Z: S \to \mathbb{R}$, $Z(\omega) = X(\omega)^2$, $Z = X^2$

$$\mathbb{E}[Z] = \sum_{\omega \in S} Z(\omega)\cdot\Pr[\omega] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 9 + \frac{1}{6}\cdot 16 + \frac{1}{6}\cdot 25 + \frac{1}{6}\cdot 36 = \frac{91}{6}$$

$$\mathbb{E}[Z] \neq \mathbb{E}[X]^2$$

Def. (Indicator random variable)

Random whose range is $\{0,1\}$.
(Usually associated with an event).

$\mathbb{1}_E: S \to \{0,1\}$ $\mathbb{1}_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{o.w.} \end{cases}$

$S = \{1,2,3,4,5,6\}$, $E = \{1,2,3\}$

$\mathbb{1}_E(\omega) = \begin{cases} 1 & \text{if } \omega = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$

$$\mathbb{E}[\mathbb{1}_E] = \sum_{\omega \in S} \mathbb{1}(\omega)\Pr[\omega] = \sum_{\omega \in E} \Pr[\omega] = \Pr[E]$$

Ex. $S = [6]$, $E = \{1,2,3\}$

$\mathbb{E}[\mathbb{1}_E] = \Pr[1] + \Pr[2] + \Pr[3] = \Pr[E]$.

Linearity of Expectation, S, $\Pr$, $R_1: S \to \mathbb{R}$, $R_2: S \to \mathbb{R}$, $R_1 + R_2$, $R_1 + R_2(\omega) = R_1(\omega) + R_2(\omega)$

Thm. $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$

Proof.
$$\begin{aligned}
\mathbb{E}[R_1 + R_2] &= \sum_{\omega \in S} R_1 + R_2(\omega)\cdot\Pr[\omega] && \text{def. } \mathbb{E} \\
&= \sum_{\omega \in S} (R_1(\omega) + R_2(\omega))\cdot\Pr[\omega] && \text{def. } R_1 + R_2 \\
&= \sum_{\omega \in S} R_1(\omega)\Pr[\omega] + \sum_{\omega \in S} R_2(\omega)\Pr[\omega] \\
&= \mathbb{E}[R_1] + \mathbb{E}[R_2] && \text{def. } \mathbb{E}
\end{aligned}$$
$\square$

Ex. $S = \{H, T\}^{100}$, $R: S \to \mathbb{R}$, $R(\omega) = $ # of heads in ω

uniform dist.

sticky coin dist.
$\Pr[\text{all heads}] = \Pr[\text{all tails}] = \frac{1}{2}$
$\Pr[\text{everything else}] = 0$.

$$\mathbb{E}[R] = \sum_{\omega \in S} \Pr[\omega]\cdot R(\omega)$$

$\mathbb{1}_{E_i}$, E_i = event i^{th} coin is heads.
$= \{\omega \mid \omega \in S,\ \omega_i = H\}$

$$R(\omega) = \sum_{i=1}^{100} \mathbb{1}_{E_i}(\omega)$$

$$\mathbb{E}[R] = \mathbb{E}\left[\sum_{i=1}^{100} \mathbb{1}_{E_i}\right] = \sum_{i=1}^{100} \mathbb{E}[\mathbb{1}_{E_i}] = \sum_{i=1}^{100} \Pr[E_i] = \sum_{i=1}^{100} \frac{1}{2} = 50$$

both sticky & uniform

generally, $E_1, E_2, \ldots, E_n$, events
$R: S \to \mathbb{R}$, $R(\omega) = $ # of events ω is in

$$\mathbb{E}[R] = \sum_{i=1}^{n} \Pr[E_i].$$