October 14

Wednesday, October 14, 2020 3:48 PM

Comp Sci 212

1. Conditional Probability
2. Independence
3. Pairwise Independence

Announcements

- Getting to know you meetings
- Midterm October 19
  - Google form if taking makeup
  - Submission by 5:15 (9:05 makeup)
  - Typed version
- No homework this week!

Def. $\Pr[A \mid B] = \dfrac{\Pr[A \cap B]}{\Pr[B]}$

Probability of A given B

Law of total probability

Thm If $E_1, \ldots, E_n$ are disjoint events, then $\sum_{i=1}^{n} \Pr[A \mid E_i]\Pr[E_i] = \Pr[A \cap (E_1 \cup \cdots \cup E_n)]$.

$E_1 \cup \cdots \cup E_n = S$

Proof. $\sum_{i=1}^{n} \Pr[A \mid E_i]\Pr[E_i] = \sum_{i=1}^{n} \Pr[A \cap E_i]$

$= \sum_{i=1}^{n} \sum_{w \in A \cap E_i} \Pr[w]$ Because E_i are disjoint, each outcome appears at most once

$= \sum_{w \in (A \cap (E_1 \cup E_2 \cup \cdots E_n))} \Pr[w]$

$= \Pr[A \cap (E_1 \cup \cdots \cup E_n)]$ □

Ex. Roll two die, what's the probability that the sum is 4?

$S = [6] \times [6]$, uniform

E_i = probability that 1st die is i

A = probability that the sum is 4

want to calculate

$\Pr[A] = \Pr[A \cap (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6)]$

$= \sum_{i=1}^{6} \Pr[A \mid E_i] \cdot \Pr[E_i]$

$\Pr[E_i] = \frac{1}{6}$ for all i.

$\Pr[A \mid E_i] = \begin{cases} \frac{1}{6} & \text{if } i \le 3 \\ 0 & \text{o.w.} \end{cases}$

$= \sum_{i=1}^{3} \frac{1}{6} \cdot \frac{1}{6} = \frac{3}{36} = \frac{1}{12}$

Def. Events A & B are independent if

$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$

If $\Pr[B] \ne 0$, $\Pr[A \mid B] = \Pr[A]$.

$E_1, \ldots, E_n$ are mutually independent if for any subset $T \subseteq [n]$,

$\Pr[\bigcap_{j \in T} E_j] = \prod_{j \in T} \Pr[E_j]$

intersection of all E_j for $j \in T$ — product for every $j \in T$

- Independence is something to be proved.

Ex. $S = \{H, T\}^{100}$, E_i be event that ith coin is heads

$E_i = \{ s \mid s \in S, s_i = H\}$

Under uniform distribution, E_i are mutually independent.

$\Pr[\bigcap_{j \in T} E_j] = \dfrac{\#\text{ of sequences s.t. } s_j = H \text{ for all } j \in T}{|S|}$

T is any subset of [100]

$= \dfrac{2^{100-|T|}}{2^{100}}$ (product rule)

$= \dfrac{1}{2^{|T|}}$

$= \prod_{j \in T} \frac{1}{2}$ — $\frac{1}{2} \cdot \frac{1}{2} \cdots$ for every element of T

$= \prod_{j \in T} \dfrac{|E_j| = 2^{99}}{|S| = 2^{100}}$

$= \prod_{j \in T} \Pr[E_j]$.

Sticky coins $\Pr[\text{all heads}] = \frac{1}{2}$

$\Pr[\text{all tails}] = \frac{1}{2}$

$\Pr[\text{everything else}] = 0$.

$\Pr[E_j] = \frac{1}{2}$ for all j.

$\Pr[\bigcap_{j \in T} E_j] = \frac{1}{2} \ne \prod_{j \in T} \Pr[E_j]$

(if $|T| \ge 2$)

not mutually independent.

Pairwise independence

$E_1, \ldots, E_n$ are pairwise independent if for every i, j

$\Pr[E_i \cap E_j] = \Pr[E_i] \cdot \Pr[E_j]$

Ex $S = \{H, T\}^2$, uniform dist.

E_1 = 1st coin is heads $\{(H,H), (H,T)\}$

E_2 = 2nd coin is heads $\{(H,H), (T,H)\}$

E_3 = both coins are the same $\{(H,H), (T,T)\}$.

$\Pr[E_1] = \Pr[E_2] = \Pr[E_3] = \frac{1}{2}$

$\Pr[E_1 \cap E_2] = \Pr[E_1 \cap E_3] = \Pr[E_2 \cap E_3] = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

$\{(H,H)\}$ are pairwise ind.

$S = \{H, T\}^n$, uniform dist.

$[n] = \{1, \ldots, n\}$

R subset of $[n]$

for each $R \subseteq [n]$ $(R \ne \emptyset)$.

$E_R = \{ \omega \mid \#\text{ of } i \text{ s.t. } i \in R \text{ and } \omega_i = H \text{ is odd}\}$

$\Pr[E_R] = \dfrac{|E_R|}{2^n} = \dfrac{2^{n-|R|}\left(\binom{|R|}{1} + \binom{|R|}{3} + \binom{|R|}{5} + \cdots\right)}{2^n}$

$2^{n-|R|}$: choices for $i \notin R$

$\binom{|R|}{1} + \binom{|R|}{3} + \cdots = \binom{|R|}{0} + \binom{|R|}{2} + \cdots$

binomial theorem

$= \dfrac{2^{n-|R|} \cdot 2^{|R|-1}}{2^n} = \dfrac{2^{n-1}}{2^n} = \dfrac{1}{2}$

$\Pr[E_R \cap E_T] = \frac{1}{4}$, E_R are pairwise ind.

Size sample space 2^n

$2^n - 1$ pairwise ind. events

n mutually ind. events

$\Pr[E_1 \cap E_2] = \Pr[E_1 \cap E_3] = \Pr[E_2 \cap E_3] = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

$\{(H, H)\}$ are pairwise ind.

$\Pr[E_1 \cap E_2 \cap E_3] = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ not mutually ind.

$\Pr[E_R \cap E_T] = \frac{1}{4}$, E_R are pairwise ind.

$R \setminus T$,	$T \setminus R$,	$R \cap T$	
even	even	odd	$= \Pr[(S \setminus E_{R \setminus T}) \cap (S \setminus E_{T \setminus R}) \cap (E_{R \cap T})] = \frac{1}{8}$
odd	odd	even	$= \Pr[E_{R \setminus T} \cap E_{T \setminus R} \cap (S \setminus E_{R \cap T})] = \frac{1}{8}$