October 12

Monday, October 12, 2020 4:05 PM

Comp Sci 212

1. Probability
2. Conditional Probability

Announcements

- Midterm October 19
- Practice midterm, 5b - can ignore

Probability

Def. Sample space - non-empty set S

$w \in S$ outcomes (elements)

$E \subseteq S$, event (subsets)

Ex. $S = \{1, 2, 3, 4, 5, 6\}$

$S = \{\text{Heads, Tails}\}$

Def. Probability function - total function $Pr: S \to \mathbb{R}$ s.t.

- $Pr[w] \geq 0$ for $w \in S$.
- $\sum_{w \in S} Pr[w] = 1$.

(sum over all outcomes in S.)

Ex. $S = \{1, 2, 3, 4, 5, 6\}$

w	$Pr[w]$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

(*)

w	$Pr[w]$
1	$\frac{1}{2}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{10}$

If $E \subseteq S$, $Pr[E] = \sum_{w \in E} Pr[w]$

(sum over all outcomes in E.)

Ex. $E = \{1, 2, 3\}$

$Pr[E] = Pr[1] + Pr[2] + Pr[3]$

$Pr[E] = \frac{1}{2} + \frac{1}{10} + \frac{1}{10} = \frac{7}{10}$

(If dist. *)

Ex. Uniform distribution - $Pr[w] = \frac{1}{|S|}$ for all w

$Pr[E] = \frac{|E|}{|S|}$

$S = \{H, T\}^{100}$, uniform distribution

↳ flipping a coin 100 times

$Pr[\text{At least 75 heads}] = \frac{|\#\text{ of ways to get} \geq 75 \text{ heads}|}{|S| = 2^{100}}$

$= |\{s \mid s \in S, s.t. \text{ s has at least 75 heads}\}|$ (generalized product rule)

Sticky coin

$S = \{H, T\}^{100}$

$Pr[\text{all heads}] = \frac{1}{2}$

$Pr[\text{all tails}] = \frac{1}{2}$

$Pr[\text{everything else}] = 0$

$Pr[\text{At least 75 heads}] = \frac{1}{2}$.

Probability rules

- Sum, if $E_1, \ldots, E_n$ are disjoint $Pr[\bigcup_{i=1}^{n} E_n] = \sum_{i=1}^{n} Pr[E_i]$ ($\bigcup_{i=1}^{n}$ union of all sets)
- Complement $Pr[S \setminus E] = 1 - Pr[E]$
- Difference $Pr[A \setminus B] = Pr[A] - Pr[A \cap B]$
- If $A \subseteq B$, $Pr[A] \leq Pr[B]$
- Inclusion-Exclusion $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$
- Union bound $Pr[A \cup B] \leq Pr[A] + Pr[B]$

$Pr[A_1 \cup A_2 \cdots A_n] \leq \sum_{i=1}^{n} Pr[A_i]$.

Thm. (Union bound) $Pr[A \cup B] \leq Pr[A] + Pr[B]$.

Proof. $Pr[A \cup B] = \sum_{w \in A \cup B} Pr[w]$

(every $w \in A \cup B$ is either in A or B, get counted at least once) $\leq \sum_{w \in A} Pr[w] + \sum_{w \in B} Pr[w]$

$= Pr[A] + Pr[B]$ □

Conditional probability

Def. $Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$.

$S = [6]^2$, uniform dist (2 die rolls).

Given that the sum is 4, what is the probability that the 1st element (or die roll) is 1?

B = set of outcomes whose sum is 4.

$= \{(1,3), (2,2), (3,1)\}$

A = set of outcomes w/ 1st die roll 1. $= \{(1,3)\}$.

$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$

Ex. $S = [100]^{25}$ sequences of length 25

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uniform dist. throwing 100-sided die 25 times.

$\Pr[\text{At least 1 die roll is 1}] = \Pr[A_1 \cup A_2 \cup A_3 \cdots A_{25}] \leq \sum_{i=1}^{25} \Pr[A_i]$ (union bound)

A_i = event i^{th} roll is 1, $= \{x \mid x \in S, x_i = 1\}$

$= 25 \cdot \frac{1}{100} = \frac{1}{4}$.

$|A_i| = 100^{24}$, $|S| = 100^{25}$, $\Pr[A_i] = \frac{1}{100}$

product rule, product rule

$|A_1| = 100^{24}$

A_1 set of sequences of length 25 where 1st element 1

1 choice for 1st element	
100	2nd
100	3rd
⋮	⋮
100	25

$\{(1,1), (1,2), (1,3)$
$(1,4), (1,5), (1,6)\}$

$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{|A \cap B|}{|B|} = \frac{1}{3}$

$= \{(1,1), (1,2), (1,3)\}$.

$\Pr[A \mid B]$: probability of event A, given event B

Given that a person has two kids, one is a boy, what is the probability that both are boys?

$S = \{(b,b), (b,g), (g,b), (g,g)\}$

uniform dist.

A = both are boys, $= \{(b,b)\}$

B = one is a boy $= \{(b,b), (b,g), (g,b)\}$

$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$.